PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Variational principles of nonlinear dynamical fluid–solid interaction systems

J. T. Xing and W. G. Price

Phil. Trans. R. Soc. Lond. A 1997 **355**, 1063-1095
doi: 10.1098/rsta.1997.0053

Variational principles of nonlinear dynamical fluid–solid interaction systems

By J. T. Xing[1,2] and W. G. Price[2]

[1] *Solid Mechanics Research Center, Beijing University of Aeronautics and Astronautics, Beijing 100083, People's Republic of China*

[2] *Department of Ship Science, University of Southampton, Highfield, Southampton SO17 1BJ, UK*

Contents

Based on the fundamental equations of continuum mechanics, the concept of Hamilton's principle and the adoption of Eulerian and Lagrangian descriptions of fluid and solid, respectively, variational principles admitting variable boundary conditions are developed to model mathematically the nonlinear dynamical behaviour of the responses and interactions between fluid and solid. The nonlinearity of the fluid is introduced through nonlinear field equations and nonlinear boundary conditions on the free surface and fluid–solid interaction interface. The structure is treated as a nonlinear elastic body. This model assumes the fluid inviscid, incompressible or

Phil. Trans. R. Soc. Lond. A (1997) **355**, 1063–1095
Printed in Great Britain

compressible and the fluid motion irrotational or rotational but isentropic along the flow path of each fluid particle. The stationary conditions of the variational principles include the governing equations of nonlinear elastic dynamics, fluid dynamics and those relating to the fluid–structure interaction interface as well as the imposed boundary conditions. A family of variational principles are obtained depending on the assumptions introduced into the mathematical model (i.e. fluid incompressible, motion irrotational, etc.) and these provide a foundation to construct numerical schemes of study to assess the dynamical behaviour of nonlinear fluid–solid interaction systems. Two simple illustrative examples are presented demonstrating the applicability of the proposed theoretical approach.

1. Introduction

Fluid–structure interaction problems in engineering involve inter-disciplinary studies relating to the fluid, flexible structure and their physical coupling mechanisms. The assessment of the dynamical behaviour of the elastic structure and fluid requires the formulation of a mathematical model representing the interactive mechanisms within the continuum and the formulation of numerical analyses to evaluate the characteristics of the dynamic system. In this paper this is achieved by the creation of a theoretical model based on the fundamental principles of continuum mechanics, the concept of Hamilton's principle and through the development of variational principles suitable for nonlinear fluid–structure interaction problems. These variational principles provide a mechanism for the transformation of the partial differential equations governing the dynamics of a structure, fluid or fluid–structure interaction system, defined by an appropriate set of physical variables (i.e. displacement, pressure, stress, etc), into an alternative set of ordinary differential or algebraic equations amenable to numerical analysis and hence, a numerical scheme of study. In this context, the Galerkin method or, more generally, the weighted residual method provides an alternative enabling role in the derivation of solutions to complex dynamical system problems (see, for example, Oden 1972; Zienkiewicz & Taylor 1989, 1991).

For fluid mechanics type problems, investigations of variational principles resembling Hamilton's principle have been undertaken by Serrin (1959), Luke (1967), Seliger & Whitham (1968) and Miles (1977). Luke incorporated a variable boundary into a variational principle developed to generate the governing equations and to analyse the behaviour of gravity waves in a two-dimensional incompressible fluid. Seliger & Whitham did not examine the implications of variable boundaries but concluded that the fluid pressure variable was the Lagrangian function within the fluid variational principle. Ikegawa & Washizu (1973), utilizing the stream function, introduced a variational principle to model an incompressible gravity flow with a free surface using the finite element method whereas, Ecer *et al.* (1983), Ecer & Akay (1983) and Ward *et al.* (1988) developed variational approaches to model incompressible, viscous fluid flows.

For linear fluid–solid interaction problems, several different forms of variational principles have been developed successfully to describe the dynamical behaviour of rigid or flexible structures and a fluid with or without free surface (see, for example, Xing 1984, 1986, 1988; Liu & Uras 1988; Xing & Price 1991; Bathe *et al.*

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

1995; Morand & Ohayon 1995; Xing *et al.* 1996). For nonlinear fluid–solid interaction problems, because of the complexities of the interactive mechanisms involved, only limited advances have been achieved in deriving descriptions of the nonlinear dynamical behaviour of both solid and fluid. Kock & Olson (1991) presented a finite element method to analyse linear and nonlinear fluid–structure interaction problems by adopting a variational indicator approach based on Hamilton's principle for inviscid, irrotational and isentropic fluid flows throughout the fluid domain. In this analysis, the entropy s is assumed constant in the whole fluid domain (rather than a function of the space coordinate as assumed herein) and two Lagrangian multipliers are introduced to make the global conservation of mass and the equation of continuity valid at the same time.

In the construction of a variational description of the dynamical behaviour of a nonlinear fluid–solid interacting system, two fundamental difficulties are encountered. The first concerns the concept of local (or space) variation and material variation. In an *Eulerian description of the fluid field*, all variables are functions of local coordinates fixed in space and time whereas in the *Lagrangian description of the structure* the motion variables are functions of the material coordinates fixed to each particle or element of the structure and time. Thus when the structure moves, the material coordinates also move from their original positions to new positions in space. These differences are further extenuated when examining time differentials, time integrals, etc. and therefore two kinds of arguments relating to fluid and structure must be included in any proposed variational functional. The second difficulty relates to variational principles involving variable boundaries. For example, in a linear analysis with all motions assumed very small, the boundary of the fluid domain during motion is assumed to be the same as the original boundary in stationary equilibrium. In a nonlinear study involving large disturbances and a free fluid surface, such an assumption is invalid and a variable boundary fluid domain must be included in the mathematical model.

In this paper, variational principles to describe the nonlinear behaviour of fluid–structure interaction systems are developed by constructing a unifying theory based on the studies of Gelfand & Fomin (1963) and McIver (1973) relating to the conceptual difficulties previously discussed, variational principles and their applications in fluid dynamics (Serrin 1959; Seliger & Whitham 1968; Miloh 1984; Rainey 1989; van Daalen *et al.* 1992; Galper & Miloh 1995) and those for nonlinear solid mechanics (Green & Zerna 1954; Oden & Reddy 1976; Washizu 1982; Xing & Price 1996). It is assumed that the fluid is inviscid, incompressible or compressible, with or without a free surface and its flow can be rotational or irrotational but isentropic along the path of each fluid particle. The solid structure is treated as a nonlinear elastic body.

The stationary conditions of the variational principles formulated include the governing equations of nonlinear elastic dynamics, consistent relationships of motions and equilibrium conditions on the fluid–solid interaction interface, the governing equations for fluid dynamics and the associated boundary conditions, including those associated with nonlinear free surface disturbances. These variational principles provide the base on which to develop numerical schemes of study to evaluate the nonlinear dynamical behaviour of the typical fluid–structure systems illustrated in figures 2–4. To demonstrate the applicability of the variational principles, by way of example, two simple dynamical problems are investigated.

2. Description of the motion of a continuum

In order to describe the motion and dynamic characteristics of a continuum (elastic solid or fluid) in a three-dimensional space, suitable systems of reference are needed (see, for example, Green & Adkins 1960; Truesdell 1966; Malvern 1970; Fung 1977). To this end, the spatial coordinate system adopted is a fixed rectangular Cartesian frame of reference with coordinate x_i $(i = 1, 2, 3)$. At time $t = t_1$, a material particle located at $x_i = X_i$ is identified by a set of ordered real numbers (X_1, X_2, X_3), referred to as the material coordinates. Since this is a symbolic coordinate used to identify a material particle, it can be chosen in different ways. For example, this role may be played by a field function $\alpha(\boldsymbol{x}, t)$ representing a particular physical quantity of the continuum. As time proceeds and the material particle moves from location to location in the three-dimensional space, its history of motion can be represented by the equation

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\boldsymbol{X}, t). \tag{2.1}$$

Mathematically, this equation defines a transformation of a domain $\Omega_1(\boldsymbol{X}, t_1)$ into a domain $\Omega_t(\boldsymbol{X}, t)$, treating time t as a parameter. It is assumed that an unique inverse of this equation exists and the Jacobian J of the transformation is positive, i.e.

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\boldsymbol{x}, t) \tag{2.2}$$

and

$$J = |\partial x_i / \partial X_j| > 0. \tag{2.3}$$

If such an equation (2.1) or (2.2) is known for every particle in the continuum, then the history of motion of the continuum is defined. In this paper, this material coordinate description is used to describe the motion of the elastic solid. The displacement, velocity and acceleration of each particle in the elastic body are therefore a function of (X_i, t) and they take the following forms, respectively:

$$U_i(\boldsymbol{X}, t) = x_i - X_i, \tag{2.4}$$

$$V_i(\boldsymbol{X}, t) = \left.\frac{\partial x_i}{\partial t}\right|_{\boldsymbol{X}} = \frac{\mathrm{D}x_i}{\mathrm{D}t} = U_{i,t}, \tag{2.5}$$

$$W_i(\boldsymbol{X}, t) = V_{i,t} = U_{i,tt}. \tag{2.6}$$

When describing the fluid flow, it is not necessary to identify the location of every fluid particle during motion but rather the instantaneous velocity field and its evolution with time. This leads to a spatial description in which the location $\boldsymbol{x}$ and the time t are taken as independent variables and the instantaneous velocity field of the fluid is represented by $v_i(\boldsymbol{x}, t)$. By applying the material derivative definition to the field function (), i.e.

$$\frac{\mathrm{D}()}{\mathrm{D}t} = ()_{,t} + v_i()_{,i}, \tag{2.7}$$

the instantaneous acceleration field is given by

$$w_i(\boldsymbol{x}, t) = \frac{\mathrm{D}v_i(\boldsymbol{x}, t)}{\mathrm{D}t} = v_{i,t} + v_j v_{i,j} = \frac{\partial v_i(\boldsymbol{X}, t)}{\partial t}. \tag{2.8}$$

(a) *The translation velocity and transmission velocity of a curved surface in space*

Let us consider a curved surface in space represented by the equation

$$f(x_1, x_2, x_3, t) = f(\boldsymbol{x}, t) = 0, \tag{2.9}$$

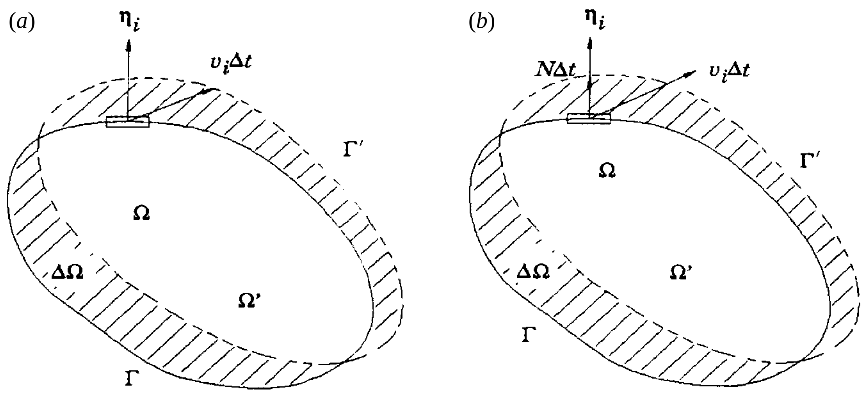

Figure 1. Continuous change of the boundary of a moving region: (*a*) the case of $\theta = 0$, i.e. a material region and $N = v_i\eta_i$; (*b*) the case of $\theta \neq 0$ and the translation velocity $N \neq v_i\eta_i$.

where $f(\boldsymbol{x}, t)$ is a continuously differentiable function. The differential of the function $f(\boldsymbol{x}, t)$ takes the form

$$\mathrm{d}f = f_{,t}\,\mathrm{d}t + f_{,i}\,\mathrm{d}x_i \tag{2.10}$$

and therefore

$$f_{,t}\,\mathrm{d}t + |\operatorname{grad} f|\,\mathrm{d}r = 0. \tag{2.11}$$

Here, $\mathrm{d}r = \mathrm{d}x_i\eta_i$ represents the projection of the elemental length $\mathrm{d}x_i$ onto the normal vector η_i of the curved surface, where

$$\eta_i = \frac{f_{,i}}{|\operatorname{grad} f|}. \tag{2.12}$$

From equation (2.11), the *translation velocity* of the curved surface is defined by

$$N = \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{f_{,t}}{|\operatorname{grad} f|}. \tag{2.13}$$

and the projection of the velocity v_i of the continuum onto the normal vector η_i of the surface takes the form

$$v_\eta = v_i\eta_i = \frac{v_i f_{,i}}{|\operatorname{grad} f|}. \tag{2.14}$$

From these results, the *transmission velocity* of the curved surface is defined by the relative velocity

$$\theta = N - v_\eta = -\frac{\mathrm{D}f/\mathrm{D}t}{|\operatorname{grad} f|}. \tag{2.15}$$

Physically, the translation velocity N of a curved surface is the velocity observed by an observer standing on the fixed reference coordinate system, but the transmission velocity θ represents the velocity observed by one standing on the material particle of the continuum with flow velocity v_i. Therefore, if $\theta = 0$, this moving curved surface is a material surface and if $\theta = -v_\eta$, it reduces to a fixed surface in space.

(*b*) *The time derivative of an integral over a moving volume in space*

It is assumed that equation (2.9) represents a convex regular region $\Omega(\boldsymbol{x}, t)$ bounded by a surface $\Gamma(\boldsymbol{x}, t)$, consisting of a finite number of parts whose outer normals form a continuous vector field, and that all regions of the solid and fluid are treated as regular. Let $F(\boldsymbol{x}, t)$ represent any continuously differentiable function in $\Omega(\boldsymbol{x}, t)$ and

$$I(t) = \int_{\Omega(\boldsymbol{x},t)} F(\boldsymbol{x}, t)\,\mathrm{d}\Omega \tag{2.16}$$

denotes the volume integral of this function at time t. The function $I(t)$ retains dependence on t because both the integrand $F(\boldsymbol{x}, t)$ and the domain $\Omega(\boldsymbol{x}, t)$ are intrinsic functions of this parameter. As t varies, $I(t)$ also varies and therefore there exists the time derivative $\mathrm{d}I/\mathrm{d}t$. In visualising the evaluation of this quantity (see figure 1), the boundary Γ of the region Ω at instant t translates with velocity N to the neighbouring surface Γ' of the region Ω' at instant $t + \Delta t$. Thus in time Δt, the change in distance $N\Delta t$ produces an elemental change in volume $\mathrm{d}\Omega = N\Delta t\,\mathrm{d}\Gamma$. Therefore, the time derivative of I is defined as

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\int_{\Omega'} F(\boldsymbol{x}, t+\Delta t)\,\mathrm{d}\Omega - \int_{\Omega} F(\boldsymbol{x}, t)\,\mathrm{d}\Omega\right] \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{\int_{\Omega} [F(\boldsymbol{x}, t+\Delta t) - F(\boldsymbol{x}, t)]\,\mathrm{d}\Omega + \int_{\Delta\Omega} F(\boldsymbol{x}, t+\Delta t)\,\mathrm{d}\Omega\right\} \\
&= \int_{\Omega} F_{,t}\,\mathrm{d}\Omega + \lim_{\Delta t\to 0} \frac{1}{\Delta t}\int_{\Gamma} F(\boldsymbol{x}, t+\Delta t) N\Delta t\,\mathrm{d}\Gamma \\
&= \int_{\Omega} F_{,t}\,\mathrm{d}\Omega + \int_{\Gamma} F(\boldsymbol{x}, t) N\,\mathrm{d}\Gamma.
\end{aligned} \tag{2.17}$$

From equations (2.13) and (2.14), it follows that $\mathrm{d}I/\mathrm{d}t$ can be rewritten as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{\Omega} F_{,t}\,\mathrm{d}\Omega + \int_{\Gamma} F v_i \eta_i\,\mathrm{d}\Gamma + \int_{\Gamma} F\theta\,\mathrm{d}\Gamma, \tag{2.18}$$

defined at time t. If the transmission velocity $\theta = 0$, the domain Ω is the material domain Ω_{M} and the time derivative of $I(t)$ reduces to the material derivative of the volume integral over the material domain. That is,

$$\frac{\mathrm{D}I}{\mathrm{D}t} = \int_{\Omega_{\mathrm{M}}} F_{,t}\,\mathrm{d}\Omega + \int_{\Gamma_{\mathrm{M}}} F v_i \eta_i\,\mathrm{d}\Gamma = \int_{\Omega_{\mathrm{M}}} \left[\frac{\mathrm{D}F}{\mathrm{D}t} + F v_{i,i}\right]\mathrm{d}\Omega, \tag{2.19}$$

after applying Green's theorem. From this result and subject to the continuum obeying the continuity equation (see equation (3.14)), it follows that (see, for example, Malvern 1970)

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\Omega_{\mathrm{M}}} \rho F\,\mathrm{d}\Omega = \int_{\Omega_{\mathrm{M}}} \rho\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}\Omega. \tag{2.20}$$

If the transmission velocity $\theta = -v_\eta = -v_i\eta_i$, the domain Ω reduces to the fixed domain Ω_{F} in space and the time derivative of $I(t)$ reduces to the form

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{\Omega_{\mathrm{F}}} F_{,t}\,\mathrm{d}\Omega. \tag{2.21}$$

(c) *A local variation and a material variation*

Let $\delta\boldsymbol{x} = \delta\boldsymbol{u}(\boldsymbol{X}, t) = \delta\boldsymbol{u}(\boldsymbol{x}, t)$ represent a virtual displacement of the particle $\boldsymbol{X}$ in the continuum from its instantaneous position $\boldsymbol{x}$. This perturbation is produced, say, by an arbitrary small additional internal or external force. The vector function $\delta\boldsymbol{u}$ is assumed to be finite valued and continuously differentiable; moreover, it conforms to any restrictions placed on the continuum position (e.g. kinematic constraints, etc.). Due to the small displacement $\delta\boldsymbol{x}$, a scalar or vector field denoted by $\phi = \phi(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ changes to $\phi^* = \phi^*(\boldsymbol{x}, t; \varepsilon)$ and the original particle at $\boldsymbol{x}$, which is now at the new position $\boldsymbol{x}^* = \boldsymbol{x} + \varepsilon\delta\boldsymbol{x}$, acquires a field value of $\phi^*(\boldsymbol{x}^*, t; \varepsilon)$. Here, ε is an independent variation parameter, $-1 < \varepsilon < 1$. A *local variation* $\bar{\delta}\phi$ in an Eulerian description and a *material variation* $\delta\phi$ in a Lagrangian description of the field function ϕ are defined, respectively, by Gelfand & Fomin (1963) to be

$$\bar{\delta}\phi = \left.\frac{\partial \phi^*(\boldsymbol{x}, t; \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \sim \phi^*(\boldsymbol{x}, t; \varepsilon) - \phi(\boldsymbol{x}, t; 0) \tag{2.22}$$

and

$$\delta\phi = \left.\frac{\partial \phi(\boldsymbol{X}, t; \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{D\phi^*(\boldsymbol{x}^*, t; \varepsilon)}{D\varepsilon}\right|_{\varepsilon=0} \sim \phi^*(\boldsymbol{x}^*, t; \varepsilon) - \phi(\boldsymbol{x}, t; 0). \tag{2.23}$$

Furthermore, they proved that there exists a relation between these variations of the field function ϕ in the form

$$\delta\phi = \bar{\delta}\phi + \delta x_i \phi_{,i}. \tag{2.24}$$

It is observed that $\delta\boldsymbol{x}$ is the initial velocity in a motion for which ε plays the role of time t. Hence the relation between the local and the material variations of a field function () is similar to the formulation denoted by equation (2.7) to calculate the material derivative of the velocity field $v_i(\boldsymbol{x}, t)$. That is,

$$\delta() = \bar{\delta}() + \delta x_i()_{,i}. \tag{2.25}$$

From these findings it can be shown that all local field derivatives commute but the material operators $\delta()$ and $\mathrm{D}()/\mathrm{D}t$ both relate to a particular particle. Therefore, the following exchangeable and non-exchangeable relations with respect to differential operations are valid:

$$\left.\begin{aligned} &\bar{\delta}()_{,i} = [\bar{\delta}()]_{,i}, \quad \bar{\delta}()_{,t} = [\bar{\delta}()]_{,t}, \quad \bar{\delta}\left[\frac{\mathrm{D}()}{\mathrm{D}t}\right] \neq \frac{\mathrm{D}}{\mathrm{D}t}[\bar{\delta}()], \\ &\delta()_{,i} \neq [\delta()]_{,i}, \quad \delta()_{,t} \neq [\delta()]_{,t}, \quad \delta\left[\frac{\mathrm{D}()}{\mathrm{D}t}\right] = \frac{\mathrm{D}}{\mathrm{D}t}[\delta()], \\ &\bar{\delta}\left[\frac{\mathrm{D}()}{\mathrm{D}t}\right] = \bar{\delta}[()_{,t} + v_i()_{,i}] = [\bar{\delta}()]_{,t} + ()_{,i}\bar{\delta}v_i + v_i[\bar{\delta}()]_{,i}. \end{aligned}\right\} \tag{2.26}$$

Moreover, by considering equations (2.19) and (2.20) for domain $\Omega_{\mathrm{F}}(\boldsymbol{x})$ and material domain $\Omega_{\mathrm{M}}(\boldsymbol{x}, t)$ in space, the following exchangeable relations with respect to integral operations also exist:

$$\begin{gathered} \bar{\delta}\int_{t_1}^{t_2} ()\,\mathrm{d}t = \int_{t_1}^{t_2} \bar{\delta}()\,\mathrm{d}t, \quad \bar{\delta}\int_{\Omega_{\mathrm{F}}(\boldsymbol{x})} ()\,\mathrm{d}\Omega = \int_{\Omega_{\mathrm{F}}(\boldsymbol{x})} \bar{\delta}()\,\mathrm{d}\Omega, \\ \delta\int_{t_1}^{t_2} ()\,\mathrm{d}t = \int_{t_1}^{t_2} \delta()\,\mathrm{d}t, \quad \delta\int_{\Omega_{\mathrm{M}}(\boldsymbol{x},t)} \rho()\,\mathrm{d}\Omega = \int_{\Omega_{\mathrm{M}}(\boldsymbol{x},t)} \rho\delta()\,\mathrm{d}\Omega, \\ \delta\int_{\Omega_{\mathrm{M}}(\boldsymbol{x},t)} ()\,\mathrm{d}\Omega = \int_{\Omega_{\mathrm{M}}(\boldsymbol{x},t)} \{\delta() + ()[\delta x_i]_{,i}\}\,\mathrm{d}\Omega, \end{gathered}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$\delta \int_{\Omega_{\mathrm{M}}(\boldsymbol{x},t)} ()\,\mathrm{d}\Omega(\boldsymbol{x},t) = \int_{\Omega_{\mathrm{M}}(\boldsymbol{X},t_1)} \delta[()J]\,\mathrm{d}\Omega(\boldsymbol{X},t_1),$$

$$\delta \int_{\Omega_{\mathrm{S}}(\boldsymbol{X},t_1)} ()\,\mathrm{d}\Omega(\boldsymbol{X},t_1) = \int_{\Omega_{\mathrm{S}}(\boldsymbol{X},t_1)} \delta()\,\mathrm{d}\Omega(\boldsymbol{X},t_1). \tag{2.27}$$

In this paper, a local variation is adopted within the fluid domain and a material variation is used in the solid domain. Therefore, for a function $()(\boldsymbol{X},t)$ defined in the material coordinate, there also exists the exchangeable relations

$$\delta\left[\frac{\mathrm{D}()(\boldsymbol{X},t)}{\mathrm{D}t}\right] = \delta\left[\frac{\partial()(\boldsymbol{X},t)}{\partial t}\right] = \delta\{[()(\boldsymbol{X},t)]_{,t}\} = [\delta()(\boldsymbol{X},t)]_{,t},$$

$$\delta\left[\frac{\partial()(\boldsymbol{X},t)}{\partial X_i}\right] = \delta\{[()(\boldsymbol{X},t)]_{,i}\} = [\delta()(\boldsymbol{X},t)]_{,i}. \tag{2.28}$$

(*d*) *The local variation of an integral over a moving volume in space*

Let the functional $H[\phi]$, defined over the moving region $\Omega(\boldsymbol{x},t)$ illustrated in figure 1, be expressible in the following form:

$$H[\phi] = \int_{t_1}^{t_2} \int_{\Omega(\boldsymbol{x},t)} F(\phi,\phi_{,t})\,\mathrm{d}\Omega\,\mathrm{d}t \tag{2.29}$$

where ϕ is a continuously differentiable function of $(\boldsymbol{x},t)$. The local variation of this functional is defined as

$$\bar{\delta}H = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{H[\phi+\varepsilon\bar{\delta}\phi] - H[\phi]\} \tag{2.30}$$

where ε is an arbitrary constant independent of ϕ, $\boldsymbol{x}$ and t and $\bar{\delta}\phi$ denotes any arbitrary local variation of the function $\phi(\boldsymbol{x},t)$, independent of ε, satisfying the conditions

$$\bar{\delta}\phi(t_1) = 0 = \bar{\delta}\phi(t_2). \tag{2.31}$$

It is noted that when a local variation of the functional $H[\phi]$ is taken, the boundary $\Gamma(\boldsymbol{x},t)$ of the region $\Omega(\boldsymbol{x},t)$ also experiences a variation and that the integral operation with respect to time t and the one with respect to space $\boldsymbol{x}$ are not interchangeable because the boundary $\Gamma(\boldsymbol{x},t)$ moves. The substitution of equation (2.29) into (2.30) gives the local variation of this functional $\bar{\delta}H$ in the form

$$\begin{aligned}
\bar{\delta}H &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\left\{\int_{\Omega(\boldsymbol{x}+\varepsilon\delta\boldsymbol{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\,\mathrm{d}\Omega - \int_{\Omega(\boldsymbol{x},t)} F(\phi,\phi_{,t})\,\mathrm{d}\Omega\right\}\mathrm{d}t\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\left\{\int_{\Omega(\boldsymbol{x},t)}[F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t}) - F(\phi,\phi_{,t}]\,\mathrm{d}\Omega\right.\\
&\qquad\left.+\int_{\Delta\Omega(\boldsymbol{x}+\varepsilon\delta\boldsymbol{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\,\mathrm{d}\Omega\right\}\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{\int_{\Omega(\boldsymbol{x},t)}\bar{\delta}F\,\mathrm{d}\Omega + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma(\boldsymbol{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\varepsilon\delta x_i\eta_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left\{\int_{\Omega(\boldsymbol{x},t)}\bar{\delta}F\,\mathrm{d}\Omega + \int_{\Gamma(\boldsymbol{x},t)} F(\phi,\phi_{,t})\delta x_i\eta_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t,
\end{aligned} \tag{2.32}$$

where, by comparison with figure 1, $\mathrm{d}\Omega = \varepsilon\delta x_i\eta_i\,\mathrm{d}\Gamma$. From equation (2.26), the first integral can be rewritten as

$$\int_{t_1}^{t_2}\int_{\Omega(\boldsymbol{x},t)}\bar{\delta}F\,\mathrm{d}\Omega\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\Omega}\left[\frac{\partial F}{\partial\phi}\bar{\delta}\phi + \frac{\partial F}{\partial\phi_{,t}}\bar{\delta}\phi_{,t}\right]\mathrm{d}\Omega\,\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left\{\int_{\Omega}\left[\frac{\partial F}{\partial\phi} - \left(\frac{\partial F}{\partial\phi_{,t}}\right)_{,t}\right]\bar{\delta}\phi\,\mathrm{d}\Omega + \int_{\Omega}\left(\frac{\partial F}{\partial\phi_{,t}}\bar{\delta}\phi\right)_{,t}\mathrm{d}\Omega\right\}\mathrm{d}t.$$

Using the relations expressed in equation (2.17), it follows that this equation becomes

$$\int_{t_1}^{t_2}\left\{\int_{\Omega}\left[\frac{\partial F}{\partial\phi} - \left(\frac{\partial F}{\partial\phi_{,t}}\right)_{,t}\right]\bar{\delta}\phi\,\mathrm{d}\Omega - \int_{\Gamma}\frac{\partial F}{\partial\phi_{,t}}N\bar{\delta}\phi\,\mathrm{d}\Gamma + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\frac{\partial F}{\partial\phi_{,t}}\bar{\delta}\phi\,\mathrm{d}\Omega\right\}\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left\{\int_{\Omega}\left[\frac{\partial F}{\partial\phi} - \left(\frac{\partial F}{\partial\phi_{,t}}\right)_{,t}\right]\bar{\delta}\phi\,\mathrm{d}\Omega - \int_{\Gamma}\frac{\partial F}{\partial\phi_{,t}}N\bar{\delta}\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t + \left\{\int_{\Omega}\frac{\partial F}{\partial\phi_{,t}}\bar{\delta}\phi\,\mathrm{d}\Omega\right\}\Bigg|_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2}\left\{\int_{\Omega(\boldsymbol{x},t)}\left[\frac{\partial F}{\partial\phi} - \left(\frac{\partial F}{\partial\phi_{,t}}\right)_{,t}\right]\bar{\delta}\phi\,\mathrm{d}\Omega - \int_{\Gamma}\frac{\partial F}{\partial\phi_{,t}}N\bar{\delta}\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t, \tag{2.33}$$

after application of the time terminal conditions given in equation (2.31). This expression, now dependent on the local variation $\bar{\delta}\phi$, is used in the development of the variational principles in § 4.

3. Governing equations

Figure 2 illustrates a selection of typical fluid–structure interaction systems under investigation as well as the nomenclature adopted. The solid body is treated as a nonlinear elastic structure and the fluid is assumed compressible, inviscid with motion isentropic along the path of each fluid particle. To assess the dynamical behaviour of a nonlinear coupled system, it is necessary to model mathematically the dynamic characteristics of the flexible structure within the solid domain Ω_{S}, the fluid with free surface in fluid domain Ω_{f} and the interacting mechanism at the fluid–structure interface Σ. This is achieved by adopting the governing equations of continuum mechanics and these are expressed in tensor notation as follows.

(*a*) *Solid domain*

In a Lagrangian description of the motions of an elastic structure, a material variation formulation is adopted. Therefore, the variables describing the dynamical behaviour, e.g. displacement U_i, momentum P_i, stress σ_{ij}, etc., are functions of the material coordinates X_i fixed to each particle of the structure and time t. The equations governing the motions of the flexible structure are (see, for example, Green & Zerna 1954; Washizu 1982)

(1) Dynamic equation,

$$\tau_{ij,j} + \hat{F}_i = P_{i,t}, \quad (X_i,t)\epsilon\Omega_{\mathrm{S}}\times(t_1,t_2), \tag{3.1}$$

where the Piola stress tensor

$$\tau_{ij} = (\delta_{ik} + U_{i,k})\sigma_{kj}, \quad (X_i,t)\epsilon\Omega_{\mathrm{S}}\times(t_1,t_2). \tag{3.2}$$

(2) Strain–displacement and velocity–displacement relations,

$$E_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}), \quad (X_i,t)\epsilon\Omega_{\mathrm{S}}\times(t_1,t_2), \tag{3.3}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

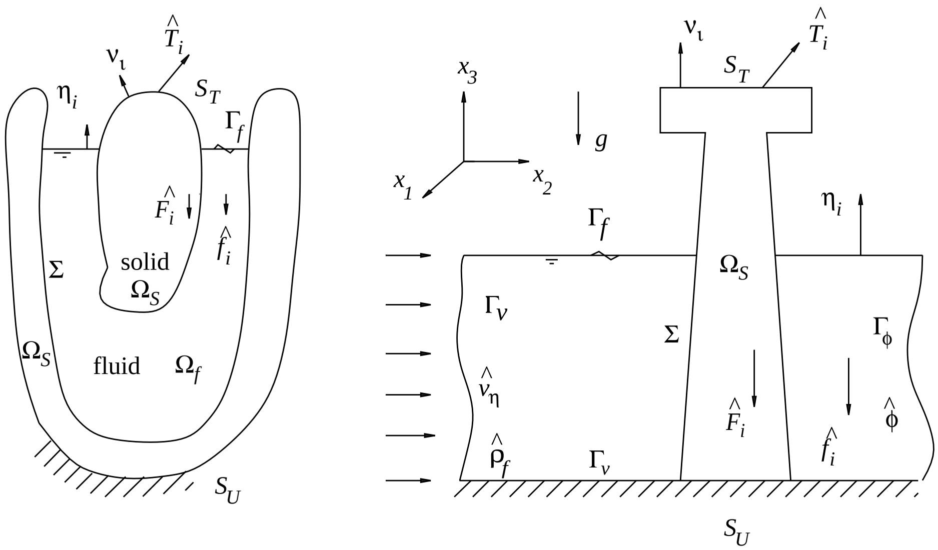

Figure 2. Fluid–structure interaction systems.

$$V_i = U_{i,t}, \quad (X_i, t)\epsilon\Omega_{\mathrm{S}} \times (t_1, t_2). \tag{3.4}$$

(3) Constitutive equations,

$$\sigma_{ij} = \partial A/\partial E_{ij}, \quad (X_i, t)\epsilon\Omega_{\mathrm{S}} \times (t_1, t_2), \tag{3.5}$$

$$P_i = \partial B/\partial V_i, \quad (X_i, t)\epsilon\Omega_{\mathrm{S}} \times (t_1, t_2). \tag{3.6}$$

(4) Boundary conditions,

$$\text{traction}: \tau_{ij}\nu_j = \hat{T}_i, \quad (X_i, t)\epsilon S_T \times [t_1, t_2], \tag{3.7}$$

$$\text{displacement}: U_i = \hat{U}_i, \quad (X_i, t)\epsilon S_U \times [t_1, t_2]. \tag{3.8}$$

(*b*) *Fluid domain*

In an Eulerian description of the fluid field, a local or space variation is used, such that, the dynamical variables describing the behaviour of the fluid, e.g. velocity v_i, pressure p, mass density ρ_{f}, etc., are functions of the spatial coordinates x_i and time t. The equations describing the fluid motion are described below.

(i) *State equation*

The internal energy per unit mass of the fluid e is a defined function of the specific volume υ, or the density ρ_{f}, and the specific entropy s and it relates to other thermodynamic quantities by the state equation (see, for example, Serrin 1959; Seliger & Whitham 1968; Woods 1975)

$$\mathrm{d}e = T\,\mathrm{d}s - p\mathrm{d}\upsilon, \tag{3.9}$$

where $T(\rho_{\mathrm{f}}, s)$ is the temperature. The internal energy e and the specific enthalpy $\psi(s, p)$ of the fluid satisfy the Legendre transformation relation

$$e - \psi = -p\upsilon = -p/\rho_{\mathrm{f}} \tag{3.10}$$

and therefore

$$\frac{\partial e}{\partial s} = T, \quad \frac{\partial e}{\partial \rho_{\mathrm{f}}} = \frac{p}{\rho_{\mathrm{f}}^2}, \tag{3.11}$$

$$\frac{\partial \psi}{\partial s} = \frac{\partial e}{\partial s}, \quad \frac{\partial \psi}{\partial p} = \frac{1}{\rho_{\mathrm{f}}}. \tag{3.12}$$

These functions e and ψ are thermodynamic potentials measured relative to a reference state. For an ideal homogeneous gas, the pressure $p = p_0(\rho_{\mathrm{f}}/\rho_{f0})^{\gamma}$, where subscript 0 denotes an initial uniform state (see, for example, Hunter 1976). If the initial state is the reference state, then the internal energy is given by the equation

$$e = \int_{\rho_{f0}}^{\rho_{\mathrm{f}}} -p_0 \left(\frac{\rho_{\mathrm{f}}}{\rho_{f0}}\right)^{\gamma} \mathrm{d}\left(\frac{1}{\rho_{\mathrm{f}}}\right) = \frac{1}{\gamma - 1}\left(\frac{p}{\rho_{\mathrm{f}}} - \frac{p_0}{\rho_{f0}}\right). \tag{3.13}$$

(ii) *Equation of continuity*

$$\rho_{f,t} + (\rho_{\mathrm{f}} v_i)_{,i} = 0, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2). \tag{3.14}$$

(iii) *Conservation of energy*

It is assumed that motion takes place without loss of energy through the generation or transfer of heat, or, more precisely, that the specific entropy s of each fluid particle remains constant during the motion, i.e.

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2), \tag{3.15}$$

or

$$(\rho_{\mathrm{f}} s)_{,t} + (\rho_{\mathrm{f}} s v_i)_{,i} = 0, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2). \tag{3.16}$$

Here, the entropy s is treated as a function of the spatial coordinate x_i and time t within the total fluid domain and not as a constant, as assumed by Kock & Olson (1991).

(iv) *Conservation of the identity of particles*

In an Eulerian description of motion, the fluid domain is occupied by different fluid particles α at each point x_i and time t. The identity coordinate of particle α is a field function of x_i and time t. As time passes, fluid particles change their positions, but their identity coordinate α remains unchanged along the path of each particle (see Lin 1963). Therefore, each coordinate satisfies the relation (see, for example, Seliger & Whitham 1968)

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = 0, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2), \tag{3.17}$$

or

$$(\rho_{\mathrm{f}} \alpha)_{,t} + (\rho_{\mathrm{f}} \alpha v_i)_{,i} = 0, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2). \tag{3.18}$$

(v) *Dynamic equation*

$$-\frac{p_{,i}}{\rho_{\mathrm{f}}} + \hat{f}_i = \frac{\mathrm{D}v_i}{\mathrm{D}t}, \quad (x_i, t)\epsilon \Omega_{\mathrm{f}} \times (t_1, t_2), \tag{3.19}$$

where, for a gravitational body force,

$$\hat{f}_i = -(gx_j\delta_{3j})_{,i}, \quad (x_i, t)\epsilon\Omega_{\mathrm{f}} \times (t_1, t_2). \tag{3.20}$$

The Clebsch transformation (see, for example, Forsyth 1890; Lamb 1932) allows the velocity of the fluid to be represented by the form

$$v_i = \phi_{,i} + s\beta_{,i} + \alpha\zeta_{,i}. \tag{3.21}$$

The quantities β, ζ and α are not uniquely defined by this expression, since any perfect differential may be added to ϕ with consequent changes in β, α and ζ. The vorticity ω_i of the velocity field v_i is given by

$$\omega_i = e_{ijk}v_{k,j} = e_{ijk}s_{,j}\beta_{,k} + e_{ijk}\alpha_{,j}\zeta_{,k}, \tag{3.22}$$

from which it follows that the field function ϕ represents a velocity potential provided the flow is irrotational, $s\beta_{,i}$ produces a vorticity caused by the distribution of the entropy s and $\alpha\zeta_{,i}$ produces a vorticity caused by the distribution of the particles in the fluid. By applying the material derivative operator in equation (2.7) to equation (3.21) and using the results of equations (3.15) and (3.17), it follows that

$$\frac{\mathrm{D}v_i}{\mathrm{D}t} = \left(\frac{\mathrm{D}\phi}{\mathrm{D}t} + s\frac{\mathrm{D}\beta}{\mathrm{D}t} + \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t} - \tfrac{1}{2}v_jv_j\right)_{,i} - s_{,i}\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha_{,i}\frac{\mathrm{D}\zeta}{\mathrm{D}t}. \tag{3.23}$$

The substitution of equations (3.20), (3.23) and equation $\psi_{,i} = p_{,i}/\rho_{\mathrm{f}} + s_{,i}\partial\psi/\partial s$, obtained from equations (3.10) and (3.12), into the dynamic equation (3.19) gives

$$\left(\tfrac{1}{2}v_jv_j - \psi - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)_{,i} = -s_{,i}\left(\frac{\mathrm{D}\beta}{\mathrm{D}t} + \frac{\partial e}{\partial s}\right) - \alpha_{,i}\frac{\mathrm{D}\zeta}{\mathrm{D}t}. \tag{3.24}$$

As noted previously, there remains a measure of arbitrariness for the chosen forms of ϕ, β, α and ζ. An examination of this equation indicates that simplifications arise if these variables satisfy the relations

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0, \quad \frac{\mathrm{D}\beta}{\mathrm{D}t} = -\frac{\partial e}{\partial s} = -\frac{\partial\psi}{\partial s} = -T, \tag{3.25}$$

where Taub (1949) refers to β as the temperature displacement. Under these conditions, it follows that

$$\left(\tfrac{1}{2}v_jv_j - \psi - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)_{,i} = \varphi_{,i} = 0, \tag{3.26}$$

or

$$\tfrac{1}{2}v_jv_j - \psi - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \lambda(t), \tag{3.27}$$

where $\lambda(t)$ represents a time dependent function. Its value depends on the reference point used to calculate the potential φ in equation (3.26). For simplicity, let $\lambda(t) = 0$. This implies that the point $\boldsymbol{x}_0$, for which $\varphi(\boldsymbol{x}_0, t) = 0$, is taken as the reference point of the integration. Thus, the dynamic equation of fluid motion takes the form

$$\tfrac{1}{2}v_jv_j - \psi - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0, \quad (x_i, t)\epsilon\Omega_{\mathrm{f}} \times (t_1, t_2). \tag{3.28}$$

For a fluid motion assumed irrotational ($s\beta_{,i} = 0 = \alpha\zeta_{,i}$), this equation reduces to

Bernoulli's equation for unsteady flow

$$\tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + gx_j\delta_{3j} = 0, \quad (x_i, t)\epsilon\Omega_{\mathrm{f}} \times (t_1, t_2), \tag{3.29}$$

where $\psi = p/\rho_{\mathrm{f}}$ and $\lambda(t) = 0$.

(vi) *Boundary conditions*

On the *free surface* it is assumed that δu_η denotes the normal component of the virtual displacement δx_i of the fluid particles such that $\delta x_i \eta_i = \delta u_\eta$. Because of the motion of the particles in the free surface, the variation δu_η is arbitrary. If an unknown equation

$$h(x_1, x_2, x_3, t) = 0, \quad (x_i, t)\epsilon\Gamma_{\mathrm{f}} \times [t_1, t_2], \tag{3.30}$$

describes the motion of the free surface, it follows that $\mathrm{D}h/\mathrm{D}t = 0$ because it is a material surface. This implies, from equations (2.12)–(2.15), that $\theta = 0$, $N = v_i\eta_i$ and the kinematic condition on the free surface is given by the equation

$$N = -\frac{h_{,t}}{|\,\mathrm{grad}\, h|} = v_i\eta_i = v_i\frac{h_{,i}}{|\,\mathrm{grad}\, h|}, \quad (x_i, t)\epsilon\Gamma_{\mathrm{f}} \times [t_1, t_2]. \tag{3.31}$$

Because the pressure on the free surface is atmospheric, $p = 0$ and, using equations (3.10), (3.28) and (3.29), the dynamic condition on the free surface is expressible in the form

$$\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j - e - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right) = 0, \quad (x_i, t)\epsilon\Gamma_{\mathrm{f}} \times [t_1, t_2], \tag{3.32}$$

for the fluid motion assumed rotational and, in the reduced form,

$$\rho_{\mathrm{f}}(e + \tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + gx_j\delta_{3j}) = 0, \quad (x_i, t)\epsilon\Gamma_{\mathrm{f}} \times [t_1, t_2], \tag{3.33}$$

for the irrotational case.

On the *boundary* Γ_v, for the fluid motion assumed rotational,

$$\rho_{\mathrm{f}}v_i\eta_i = \hat{\rho}_{\mathrm{f}}\hat{v}_\eta, \quad (x_i, t)\epsilon\Gamma_v \times [t_1, t_2], \tag{3.34}$$

$$s = \hat{s}, \quad \alpha = \hat{\alpha}, \quad (x_i, t)\epsilon\Gamma_v \times [t_1, t_2], \tag{3.35}$$

whereas, for the irrotational case,

$$\rho_{\mathrm{f}}\phi_{,i}\eta_{,i} = \hat{\rho}_{\mathrm{f}}\hat{v}_\eta, \quad (x_i, t)\epsilon\Gamma_v \times [t_1, t_2]. \tag{3.36}$$

On the *boundary* Γ_ϕ,

$$\phi = \hat{\phi}, \quad \beta = \hat{\beta}, \quad \zeta = \hat{\zeta}, \quad (x_i, t)\epsilon\Gamma_\phi \times [t_1, t_2], \tag{3.37}$$

for both rotational and irrotational cases.

Remark 3.1. On the free surface, alternative forms of boundary condition to those expressed can be developed. By way of a simple example, let us assume the fluid is incompressible ($e = 0$, $\psi = p/\rho_{\mathrm{f}}$), the fluid motion irrotational and the free surface disturbance

$$h(x_1, x_2, x_3, t) = \eta(x_1, x_2, t) - x_3 = 0,$$

where $\eta(x_1, x_2, t)$ represents a surface wave disturbance. It follows from the kinematic condition

$$\frac{\mathrm{D}h}{\mathrm{D}t} = 0 = \frac{\mathrm{D}\eta}{\mathrm{D}t} - \frac{\mathrm{D}x_3}{\mathrm{D}t}$$

or

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = v_3 = \phi_{,i}\delta_{3i},$$

equation (3.33)

$$\eta(x_1, x_2, t) = -1/g(\phi_{,t} + \tfrac{1}{2}\phi_{,j}\phi_{,j})$$

and through the manipulation of these two equations that

$$\phi_{,tt} + 2\phi_{,i}\phi_{,ti} + \tfrac{1}{2}\phi_{,i}(\phi_{,j}\phi_{,j})_{,i} + g\phi_{,i}\delta_{3i} = 0.$$

This equation represents the nonlinear boundary condition on the free surface, which, on neglect of products of terms, reduces to the usual form of the linear surface boundary condition with $\eta = -\phi_{,t}/g$.

(c) *Fluid–structure interface*

Let us assume that there exists no discontinuity on the fluid–structure interaction interface Σ during motions and the variation process. This implies that both the virtual displacement δx_i of the fluid and the virtual displacement δU_i of the solid have the same normal component at each point $x_i = X_i + U_i$ on the interaction boundary Σ (i.e. $\delta x_i\eta_i = -\delta U_i\nu_i = -\delta U_\nu$) and that the translation velocity of the boundary Σ in the fluid domain equals the normal velocity of the solid on the boundary Σ (i.e. $N = V_i\eta_i$). Therefore, the motion on the fluid–structure interaction interface Σ satisfies the following imposed conditions on the velocity and pressure.

For fluid motion assumed rotational, the normal velocity satisfies the relation

$$v_i\eta_i = V_i\eta_i = -V_i\nu_i, \quad (x_i, t)\epsilon\Sigma \times [t_1, t_2], \tag{3.38}$$

whereas, in the irrotational case,

$$\phi_{,i}\eta_i = V_i\eta_i = -V_i\nu_i, \quad (x_i, t)\epsilon\Sigma \times [t_1, t_2]. \tag{3.39}$$

For the rotational case, the pressure satisfies the interface condition

$$\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j - e - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right) + \nu_i\tau_{ij}\nu_j = 0, \quad (x_i, t)\epsilon\Sigma \times [t_1, t_2], \tag{3.40}$$

and, for the irrotational case,

$$\rho_{\mathrm{f}}(e + gx_j\delta_{3j} + \phi_{,t} + \tfrac{1}{2}\phi_{,j}\phi_{,j}) - \nu_i\tau_{ij}\nu_j = 0, \quad (x_i, t)\epsilon\Sigma \times [t_1, t_2]. \tag{3.41}$$

The tangential force satisfies the relation

$$\xi_i\tau_{ij}\nu_j = 0, \quad (x_i, t)\epsilon\Sigma \times [t_1, t_2] \tag{3.42}$$

for both rotational and irrotational cases involving an inviscid fluid.

(d) *Variational conditions at initial time t_1 and final time t_2*

The variational conditions applied at initial time t_1 and final time t_2 take the following forms. For the rotational case,

$$\bar{\delta}\phi(t_1) = 0 = \bar{\delta}\phi(t_2), \quad x_i\epsilon\hat{\Omega}_{\mathrm{f}}, \tag{3.43}$$

$$\overline{\delta}\beta(t_1) = 0 = \overline{\delta}\beta(t_2), \quad x_i \epsilon \hat{\Omega}_{\mathrm{f}}, \tag{3.44}$$
$$\overline{\delta}\zeta(t_1) = 0 = \overline{\delta}\zeta(t_2), \quad x_i \epsilon \hat{\Omega}_{\mathrm{f}}, \tag{3.45}$$
$$\delta U_i(t_1) = 0 = \delta U_i(t_2), \quad X_i \epsilon \hat{\Omega}_{\mathrm{S}}, \tag{3.46}$$

and these reduce to

$$\overline{\delta}\phi(t_1) = 0 = \overline{\delta}\phi(t_2), \quad x_i \epsilon \hat{\Omega}_{\mathrm{f}}, \tag{3.47}$$
$$\delta U_i(t_1) = 0 = \delta U_i(t_2), \quad X_i \epsilon \hat{\Omega}_{\mathrm{S}} \tag{3.48}$$

for the irrotational case.

4. Variational principles

(*a*) *Fluid motion assumed rotational*

From equation (3.22), it is found that the vorticity of the velocity field is caused by the distribution of the entropy s and the distribution of the fluid particles α. Therefore, when treating rotational motion, the equation of conservation of energy represented in (3.15) and the equation of conservation of the identity of the fluid particles in (3.17) must be considered. For example, in a simple incompressible flow with constant entropy, ($s_{,i} = 0$) assumed throughout the fluid domain, there exists the possibility of a fluid flow being rotational. However, from relation (3.17) with parameters α and ζ omitted from the mathematical model, the vorticity $\omega_i \equiv 0$ throughout the fluid domain implies a restriction to flows assumed irrotational with the exclusion of all rotational possibilities. This proved a long-standing difficulty in fluid dynamics until Lin (1963) perceived the necessity of introducing condition (3.17) into a more general mathematical model as discussed by Seliger & Whitham (1968).

(i) *Compressible fluid with $s_{,i} \not\equiv 0$ in the fluid domain Ω_{f}*

It is found that amongst all the admissible solid displacement U_i satisfying the strain–displacement relations in equation (3.3), the velocity–displacement relations in equation (3.4), the displacement boundary condition in equation (3.8) and the time instant conditions (3.46), as well as the admissible fluid field arguments ρ_{f}, v_i, ϕ, s, β, α, ζ satisfying equations (3.37), (3.43)–(3.45) and the function h describing the free surface disturbance, the actual motion satisfying the governing equations in (3.1), (3.7), (3.14), (3.16), (3.18), (3.21), (3.25), (3.28), (3.31), (3.32), (3.34), (3.35), (3.38), (3.40) and (3.42) makes the 9-argument functional

$$\begin{aligned} H_9[\rho_{\mathrm{f}}, v_i, \phi, s, \beta, \alpha, \zeta, h, U_i] &= \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \mathrm{d}\Omega \right. \\ &\quad \left. + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta (\phi + \hat{s}\beta + \hat{\alpha}\zeta)\, \mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i]\, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i\, \mathrm{d}S \right\} \mathrm{d}t \end{aligned} \tag{4.1}$$

stationary, if the constitutive relations expressed in equations (3.5), (3.6), (3.10), (3.11) and (3.12) are satisfied.

In deriving the proof of this variational principle, it is noted that the fluid domain Ω_{f} is a moving domain in space since the free surface Γ_{f} and the fluid–solid interaction boundary Σ change during motion, but the material domain Ω_{S} of the elastic structure, with respect to the material coordinates X_i, remains unchanged. For these reasons, the free surface disturbance function h appears as an argument of the functional. By taking the *local variation* of the integral over the fluid domain Ω_{f} and the *material variation* of the integral over the solid domain Ω_{S}, the variation of this functional is given by

$$
\begin{aligned}
&\delta^{(\mathrm{fs})} H_9[\rho_{\mathrm{f}}, v_i, \phi, s, \beta, \alpha, \zeta, h, U_i] \\
&= \int_{t_1}^{t_2} \bigg\{ \int_{\Omega_{\mathrm{f}}} \bar{\delta} \left[\rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s \frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \right] \mathrm{d}\Omega \\
&\quad + \int_{\Gamma_{\mathrm{f}} \cup \Sigma} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s \frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \delta x_i \eta_i \, \mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta \bar{\delta} (\phi + \hat{s}\beta + \hat{\alpha}\zeta) \, \mathrm{d}\Gamma \bigg\} \, \mathrm{d}t \\
&\quad - \int_{t_1}^{t_2} \bigg\{ \int_{\Omega_{\mathrm{S}}} \delta (A - B - U_i \hat{F}_i) \, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i \delta U_i \, \mathrm{d}S \bigg\} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \bigg\{ \int_{\Omega_{\mathrm{f}}} \bar{\delta} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - \psi - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s \frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \mathrm{d}\Omega \\
&\quad + \int_{\Omega_{\mathrm{f}}} \rho_{\mathrm{f}} \bigg[v_j \bar{\delta} v_j - \left(\frac{\partial e}{\partial s} + \frac{\mathrm{D}\beta}{\mathrm{D}t} \right) \bar{\delta} s - \bar{\delta} \left(\frac{\mathrm{D}\phi}{\mathrm{D}t} \right) \\
&\quad - s \bar{\delta} \left(\frac{\mathrm{D}\beta}{\mathrm{D}t} \right) - \frac{\mathrm{D}\zeta}{\mathrm{D}t} \bar{\delta} \alpha - \alpha \bar{\delta} \left(\frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \bigg] \mathrm{d}\Omega \\
&\quad + \int_{\Gamma_{\mathrm{f}} \cup \Sigma} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s \frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \delta x_i \eta_i \mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta (\bar{\delta}\phi + \hat{s}\bar{\delta}\beta + \hat{\alpha}\bar{\delta}\zeta) \, \mathrm{d}\Gamma \bigg\} \, \mathrm{d}t \\
&\quad - \int_{t_1}^{t_2} \bigg\{ \int_{\Omega_{\mathrm{S}}} (\sigma_{ij} \delta E_{ij} - P_i \delta V_i - \delta U_i \hat{F}_i) \, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i \delta U_i \, \mathrm{d}S \bigg\} \, \mathrm{d}t. \qquad (4.2)
\end{aligned}
$$

In deriving these relations, equations (2.27) and (2.32) are used to calculate the local variations $\bar{\delta}()$ or material variations $\delta()$, in addition to the non-variational constraint conditions given in equations (3.5), (3.6), (3.10), (3.11) and (3.12).

From equations (2.26), (3.3) and (3.4), the following additional relations are obtained:

$$
\begin{aligned}
\rho_{\mathrm{f}} \bar{\delta} \left(\frac{\mathrm{D}\phi}{\mathrm{D}t} \right) &= (\rho_{\mathrm{f}} \bar{\delta}\phi)_{,t} + (\rho_{\mathrm{f}} v_i \bar{\delta}\phi)_{,i} - [\rho_{f,t} + (\rho_{\mathrm{f}} v_i)_{,i}] \bar{\delta}\phi + \rho_{\mathrm{f}} \phi_{,i} \bar{\delta} v_i, \\
\rho_{\mathrm{f}} s \bar{\delta} \left(\frac{\mathrm{D}\beta}{\mathrm{D}t} \right) &= (\rho_{\mathrm{f}} s \bar{\delta}\beta)_{,t} + (\rho_{\mathrm{f}} s v_i \bar{\delta}\beta)_{,i} - [(\rho_{\mathrm{f}} s)_{,t} + (\rho_{\mathrm{f}} s v_i)_{,i}] \bar{\delta}\beta + \rho_{\mathrm{f}} s \beta_{,i} \bar{\delta} v_i, \\
\rho_{\mathrm{f}} \alpha \bar{\delta} \left(\frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) &= (\rho_{\mathrm{f}} \alpha \bar{\delta}\zeta)_{,t} + (\rho_{\mathrm{f}} \alpha v_i \bar{\delta}\zeta)_{,i} - [(\rho_{\mathrm{f}} \alpha)_{,t} + (\rho_{\mathrm{f}} \alpha v_i)_{,i}] \bar{\delta}\zeta + \rho_{\mathrm{f}} \alpha \zeta_{,i} \bar{\delta} v_i, \\
\sigma_{ij} \delta E_{ij} &= \tau_{ij} \delta U_{i,j}, \quad P_i \delta V_i = P_i \delta U_{i,t}. \qquad (4.3)
\end{aligned}
$$

The substitution of these results into equation (4.2), the application of Green's theorem (Σ assumed continuous) and the application of equation (2.17) to the terms $(\rho_{\mathrm{f}}\bar{\delta}\phi)_{,t}$, $(\rho_{\mathrm{f}}s\bar{\delta}\beta)_{,t}$ and $(\rho_{\mathrm{f}}\alpha\bar{\delta}\zeta)_{,t}$ associated with the moving space domain Ω_{f}, together with the substitution $\delta x_i\eta_i = \delta u_\eta$ ($\equiv \delta x_i h_{,i}/|\operatorname{grad} h|$) on the free surface Γ_{f}, allows the variation of the functional H_9 to be expressed as

$$
\begin{aligned}
&\delta^{(\mathrm{fs})}H_9[\rho_{\mathrm{f}}, v_i, \phi, s, \beta, \alpha, \zeta, h, U_i] \\
&\quad = \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\left[\bar{\delta}\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j - \psi - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)\right.\right. \\
&\qquad \times[\rho_{f,t} + (\rho_{\mathrm{f}}v_i)_{,i}]\bar{\delta}\phi + [(\rho_{\mathrm{f}}s)_{,t} + (\rho_{\mathrm{f}}sv_i)_{,i}]\bar{\delta}\beta + [(\rho_{\mathrm{f}}\alpha)_{,t} + (\rho_{\mathrm{f}}\alpha v_i)_{,i}]\bar{\delta}\zeta \\
&\qquad \left. + \rho_{\mathrm{f}}(v_i - \phi_{,i} - s\beta_{,i} - \alpha\zeta_{,i})\bar{\delta}v_i - \rho_{\mathrm{f}}\left(\frac{\partial e}{\partial s} + \frac{\mathrm{D}\beta}{\mathrm{D}t}\right)\bar{\delta}s - \rho_{\mathrm{f}}\frac{\mathrm{D}\zeta}{\mathrm{D}t}\bar{\delta}\alpha\right]\mathrm{d}\Omega \\
&\qquad + \int_{\Gamma_{\mathrm{f}}}\left[\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j - e - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)\delta u_\eta\right. \\
&\qquad \left. + \rho_{\mathrm{f}}(N - v_i\eta_i)(\bar{\delta}\phi + s\bar{\delta}\beta + \alpha\bar{\delta}\zeta)\right]\mathrm{d}\Gamma \\
&\qquad - \int_{\Gamma_\phi}\rho_{\mathrm{f}}v_i\eta_i(\bar{\delta}\phi + s\bar{\delta}\beta + \alpha\bar{\delta}\zeta)\,\mathrm{d}\Gamma \\
&\qquad \left. - \int_{\Gamma_v}[(\rho_{\mathrm{f}}v_i\eta_i - \hat{\rho}_{\mathrm{f}}\hat{v}_\eta)\bar{\delta}\phi + (\rho_{\mathrm{f}}sv_i\eta_i - \hat{\rho}_{\mathrm{f}}\hat{s}\hat{v}_\eta)\bar{\delta}\beta + (\rho_{\mathrm{f}}\alpha v_i\eta_i - \hat{\rho}_{\mathrm{f}}\hat{\alpha}\hat{v}_\eta)\bar{\delta}\zeta]\,\mathrm{d}\Gamma\right\}\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}(\tau_{ij,j} - P_{i,t} + \hat{F}_i)\delta U_i\,\mathrm{d}\Omega\right. \\
&\qquad \left. - \int_{S_T}(\tau_{ij}\nu_j - \hat{T}_i)\delta U_i\,\mathrm{d}S - \int_{S_U}\tau_{ij}\nu_j\delta U_i\,\mathrm{d}S\right\}\mathrm{d}t \\
&\qquad - \int_{t_1}^{t_2}\int_{\Sigma}\left\{\left[\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j - e - gx_j\delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right) + \nu_i\tau_{ij}\nu_j\right]\delta U_\nu\right. \\
&\qquad \left. - \xi_i\tau_{ij}\nu_j\delta U_\xi + \rho_{\mathrm{f}}(V_i\eta_i - v_i\eta_i)(\bar{\delta}\phi + s\bar{\delta}\beta + \alpha\bar{\delta}\zeta)\right\}\mathrm{d}\Sigma\,\mathrm{d}t \\
&\qquad - \left\{\int_{\Omega_{\mathrm{S}}}P_i\delta U_i\,\mathrm{d}\Omega + \int_{\Omega_{\mathrm{f}}}\rho_{\mathrm{f}}(\bar{\delta}\phi + s\bar{\delta}\beta + \alpha\bar{\delta}\zeta)\,\mathrm{d}\Omega\right\}\Bigg|_{t_1}^{t_2}.
\end{aligned} \tag{4.4}
$$

Through the variational conditions expressed in equations (3.43)–(3.46) at time instants t_1 and t_2, conditions (3.37) over the boundary Γ_ϕ and the displacement boundary condition of equation (3.8), the integrals over Γ_ϕ and S_U, as well as the last term in equation (4.4), vanish. Further, because of the independence of the variations $\bar{\delta}\phi$, $\bar{\delta}s$, $\bar{\delta}\beta$, $\bar{\delta}\alpha$, $\bar{\delta}\zeta$, $\bar{\delta}\rho_{\mathrm{f}}$, $\bar{\delta}v_i$ in the fluid domain Ω_{f}, the variations $\bar{\delta}\phi$, $\bar{\delta}\beta$, $\bar{\delta}\zeta$ over the boundaries Γ_{f}, Γ_v and Σ, the variation δu_η on the free surface Γ_{f}, the variations δU_i in the solid domain Ω_{S} and over the boundary S_T and the variations δU_ν and δU_ξ over the fluid–structure interaction surface Σ, equations (3.1), (3.7), (3.14), (3.16), (3.18), (3.21), (3.25), (3.28), (3.31), (3.32), (3.34), (3.35), (3.38), (3.40) and (3.42) result when $\delta^{(\mathrm{fs})}H_9 = 0$, and vice versa.

Remark 4.1. If the Clebsch transformation expressed in equation (3.21) is considered as a constraint condition on the functional H_9 represented in equation (4.1), the alternative functional with 8-arguments (ρ_{f}, ϕ, s, β, α, ζ, h, U_i) can be derived

$$\begin{aligned} &H_8[\rho_{\mathrm{f}}, \phi, s, \beta, \alpha, \zeta, h, U_i] \\ &\quad = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \rho_{\mathrm{f}}[-\tfrac{1}{2}(\phi_{,i} + s\beta_{,i} + \alpha\zeta_{,i})(\phi_{,i} + s\beta_{,i} + \alpha\zeta_{,i}) \right. \\ &\qquad \left. - \phi_{,t} - s\beta_{,t} - \alpha\zeta_{,t} - e - gx_j\delta_{3j}]\,\mathrm{d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}}\hat{v}_\eta(\phi + \hat{s}\beta + \hat{\alpha}\zeta)\,\mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i\,\mathrm{d}S \right\} \mathrm{d}t, \end{aligned} \tag{4.5}$$

in which the fluid velocity v_i is excluded from the variational arguments.

The variational constraint conditions on this functional are given in equations (3.3), (3.4), (3.8), (3.37), (3.21) and (3.43)–(3.46) and its stationary conditions are represented in equations (3.1), (3.7), (3.14), (3.16), (3.18), (3.25), (3.28), (3.31), (3.32), (3.34), (3.35), (3.38), (3.40) and (3.42), in which v_i and the material derivatives $\mathrm{D}()/\mathrm{D}t$ are replaced by their equivalent expressions through equations (3.21) and (2.7), respectively.

Remark 4.2. The replacement of the internal fluid energy e with the enthalpy ψ and the pressure p by expression $\psi - p/\rho_{\mathrm{f}}$ in equation (3.10) allows the functionals H_9 in (4.1) and H_8 in (4.5) to be replaced, respectively, by

$$\begin{aligned} &H_{10p}[p, \rho_{\mathrm{f}}, v_i, \phi, s, \beta, \alpha, \zeta, h, U_i] \\ &\quad = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - \psi + \frac{p}{\rho_{\mathrm{f}}} - gx_j\delta_{3j} \right. \right. \\ &\qquad \left. \left. - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \mathrm{d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}}\hat{v}_\eta(\phi + \hat{s}\beta + \hat{\alpha}\zeta)\,\mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i\,\mathrm{d}S \right\} \mathrm{d}t \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &H_{9p}[p, \rho_{\mathrm{f}}, \phi, s, \beta, \alpha, \zeta, h, U_i] \\ &\quad = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \rho_{\mathrm{f}}[-\tfrac{1}{2}(\phi_{,i} + s\beta_{,i} + \alpha\zeta_{,i})(\phi_{,i} + s\beta_{,i} + \alpha\zeta_{,i}) \right. \\ &\qquad - \phi_{,t} - s\beta_{,t} - \alpha\zeta_{,t} - \psi + (p/\rho_{\mathrm{f}}) - gx_j\delta_{3j}]\,\mathrm{d}\Omega \\ &\qquad \left. + \int_{\Gamma_v} \hat{\rho}_{\mathrm{f}}\hat{v}_\eta(\phi + \hat{s}\beta + \hat{\alpha}\zeta)\,\mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i\,\mathrm{d}S \right\} \mathrm{d}t. \end{aligned} \tag{4.7}$$

Here, the pressure p is introduced into the variational arguments and the constitutive equation represented in the second equation of (3.12) is now one of the stationary conditions. The variational constraint conditions and stationary conditions are the same as those applied to the functionals H_9 and H_8, respectively.

Remark 4.3. Equations (3.3), (3.4), (3.8), (3.37) are the variational constraint conditions applicable to the functional H_{10p} in equation (4.6). A relaxation of these conditions is achieved through use of the Lagrangian multiplier method (for example, see, Courant & Hilbert 1962). By this means, the following functional with 14-arguments (p, ρ_{f}, v_i, ϕ, s, β, α, ζ, h, U_i, σ_{ij}, E_{ij}, P_i, V_i) is obtained:

$$\begin{aligned} H_{14}&[p,\rho_{\mathrm{f}},v_i,\phi,s,\beta,\alpha,\zeta,h,U_i,\sigma_{ij},E_{ij},P_i,V_i] \\ &= \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j-\psi+\frac{p}{\rho_{\mathrm{f}}}-\ gx_j\delta_{3j}-\frac{\mathrm{D}\phi}{\mathrm{D}t}-s\frac{\mathrm{D}\beta}{\mathrm{D}t}-\alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)\mathrm{d}\Omega\right. \\ &\quad+\int_{\Gamma_v}\hat{\rho}_{\mathrm{f}}\hat{v}_\eta(\phi+\hat{s}\beta+\hat{\alpha}\zeta)\,\mathrm{d}\Gamma \\ &\quad\left.+\int_{\Gamma_\phi}\rho_{\mathrm{f}}v_i\eta_i[(\phi-\hat{\phi})+s(\beta-\hat{\beta})+\alpha(\zeta-\hat{\zeta})]\,\mathrm{d}\Gamma\right\}\mathrm{d}t \\ &\quad-\int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}[A(E_{ij})-B(V_i)-U_i\hat{F}_i+P_i(V_i-U_{i,t})\right. \\ &\quad-\sigma_{ij}(E_{ij}-\tfrac{1}{2}(U_{i,j}+U_{j,i}+U_{k,i}U_{k,j}))]\,\mathrm{d}\Omega \\ &\quad\left.-\int_{S_T}\hat{T}_iU_i\,\mathrm{d}S-\int_{S_U}\tau_{ij}\nu_j(U_i-\hat{U}_i)\,\mathrm{d}S\right\}\mathrm{d}t. \end{aligned} \tag{4.8}$$

The variational constraint conditions of this functional at times t_1 and t_2 reduce to the conditions given in equations (3.43)–(3.46). The stationary conditions of this functional include all the governing equations of fluid–structure dynamic interaction problems.

Remark 4.4. The variational constraint conditions at times t_1 and t_2 given in equations (3.43)–(3.46) can also be released, if four time conditions at times t_1 and t_2 are considered (see Xing & Price 1992).

(ii) *Compressible fluid with* $s_{,i}\equiv 0$ *throughout the fluid domain* Ω_{f}

If the entropy s of the fluid is treated as constant throughout the fluid domain Ω_{f}, equation (3.15) is automatically satisfied and $s_{,i}\equiv 0$ in Ω_{f}. From this condition, the vorticity ω_i in equation (3.22) is independent of the entropy s and the fluid field velocity v_i can be represented as

$$v_i=\phi_{,i}+\alpha\zeta_{,i}, \tag{4.9}$$

because the term $s\beta_{,i}$ in equation (3.21) can be absorbed into $\phi_{,i}$. In this case, the functionals H_9 in (4.1), H_8 in (4.5), H_{10p} in (4.6), H_{9p} in (4.7) and H_{14} in (4.8), respectively, reduce to the following forms:

$$\begin{aligned} H_7&[\rho_{\mathrm{f}},v_i,\phi,\alpha,\zeta,h,U_i] \\ &= \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j-e-gx_j\delta_{3j}-\frac{\mathrm{D}\phi}{\mathrm{D}t}-\alpha\frac{\mathrm{D}\zeta}{\mathrm{D}t}\right)\mathrm{d}\Omega\right. \\ &\quad\left.+\int_{\Gamma_v}\hat{\rho}_{\mathrm{f}}\hat{v}_\eta(\phi+\hat{\alpha}\zeta)\,\mathrm{d}\Gamma\right\}\mathrm{d}t \\ &\quad-\int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}[A(E_{ij})-B(V_i)-U_i\hat{F}_i]\,\mathrm{d}\Omega-\int_{S_T}\hat{T}_iU_i\,\mathrm{d}S\right\}\mathrm{d}t, \end{aligned} \tag{4.10}$$

$$H_6[\rho_{\rm f}, \phi, \alpha, \zeta, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm f}} \rho_{\rm f}[-\tfrac{1}{2}(\phi_{,i} + \alpha\zeta_{,i})(\phi_{,i} + \alpha\zeta_{,i}) - \phi_{,t} - \alpha\zeta_{,t} - e - gx_j\delta_{3j}]\, {\rm d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\rm f}\hat{v}_\eta(\phi + \hat{\alpha}\zeta)\, {\rm d}\Gamma \right\} {\rm d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm S}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\, {\rm d}\Omega - \int_{S_T} \hat{T}_i U_i\, {\rm d}S \right\} {\rm d}t, \tag{4.11}$$

$$H_{8p}[p, \rho_{\rm f}, v_i, \phi, \alpha, \zeta, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm f}} \rho_{\rm f} \left(\tfrac{1}{2}v_j v_j - \psi + \frac{p}{\rho_{\rm f}} - gx_j\delta_{3j} - \frac{{\rm D}\phi}{{\rm D}t} - \alpha\frac{{\rm D}\zeta}{{\rm D}t} \right) {\rm d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\rm f}\hat{v}_\eta(\phi + \hat{\alpha}\zeta)\, {\rm d}\Gamma \right\} {\rm d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm S}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\, {\rm d}\Omega - \int_{S_T} \hat{T}_i U_i\, {\rm d}S \right\} {\rm d}t, \tag{4.12}$$

$$H_{7p}[p, \rho_{\rm f}, \phi, \alpha, \zeta, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm f}} \rho_{\rm f}[-\tfrac{1}{2}(\phi_{,i} + \alpha\zeta_{,i})(\phi_{,i} + \alpha\zeta_{,i}) - \phi_{,t} - \alpha\zeta_{,t} - \psi + (p/\rho_{\rm f}) - gx_j\delta_{3j}]\, {\rm d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\rm f}\hat{v}_\eta(\phi + \hat{\alpha}\zeta)\, {\rm d}\Gamma \right\} {\rm d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm S}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\, {\rm d}\Omega - \int_{S_T} \hat{T}_i U_i\, {\rm d}S \right\} {\rm d}t, \tag{4.13}$$

and

$$H_{12}[p, \rho_{\rm f}, v_i, \phi, \alpha, \zeta, h, U_i, \sigma_{ij}, E_{ij}, P_i, V_i] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm f}} \rho_{\rm f} \left(\tfrac{1}{2}v_j v_j - \psi + \frac{p}{\rho_{\rm f}} - gx_j\delta_{3j} - \frac{{\rm D}\phi}{{\rm D}t} - \alpha\frac{{\rm D}\zeta}{{\rm D}t} \right) {\rm d}\Omega + \int_{\Gamma_v} \hat{\rho}_{\rm f}\hat{v}_\eta(\phi + \hat{\alpha}\zeta)\, {\rm d}\Gamma + \int_{\Gamma_\phi} \rho_{\rm f} v_i \eta_i[(\phi - \hat{\phi}) + \alpha(\zeta - \hat{\zeta})]\, {\rm d}\Gamma \right\} {\rm d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\rm S}} [A(E_{ij}) - B(V_i) - U_i\hat{F}_i + P_i(V_i - U_{i,t}) - \sigma_{ij}(E_{ij} - \tfrac{1}{2}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}))]\, {\rm d}\Omega - \int_{S_T} \hat{T}_i U_i\, {\rm d}S - \int_{S_U} \tau_{ij}\nu_j(U_i - \hat{U}_i)\, {\rm d}S \right\} {\rm d}t. \tag{4.14}$$

For these functionals, the applied variational constraint and stationary conditions correspond to those of the original functionals with all terms associated with entropy s and β excluded.

(iii) *Incompressible fluid*

For an incompressible fluid, the fluid density is a prescribed constant, $\rho_{\rm f} = \tilde{\rho}_{\rm f}$ (say), the variation $\bar{\delta}\tilde{\rho}_{\rm f} \equiv 0$ and the equation of continuity expressed in equation

(3.14) reduces to the form

$$v_{i,i} = 0. \tag{4.15}$$

The replacement of ρ_{f} by $\tilde{\rho}_{\mathrm{f}}$ in the expressions derived for the functionals in § 4 *a* (i) and § 4 *a* (ii) allows the corresponding functionals for an incompressible fluid to be obtained. For example, for functional H_9 in equation (4.1), it follows that

$$\begin{aligned}\tilde{H}_8&[v_i, \phi, s, \beta, \alpha, \zeta, h, U_i] \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \tilde{\rho}_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \mathrm{d}\Omega \right. \\ &\qquad \left. + \int_{\Gamma_v} \tilde{\rho}_{\mathrm{f}} \hat{v}_\eta (\phi + \hat{s}\beta + \hat{\alpha}\zeta)\, \mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i]\, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i \, \mathrm{d}S \right\} \mathrm{d}t. \end{aligned} \tag{4.16}$$

It is noted that the internal energy of an incompressible fluid is a function of the variable s only, so that $e \equiv 0$ if $s_{,i} \equiv 0$ throughout the fluid domain Ω_{f}. Therefore, for $\rho_{\mathrm{f}} = \tilde{\rho}_{\mathrm{f}}$ and $s_{,i} \equiv 0$, the terms e and $\psi - p/\rho_{\mathrm{f}}$ in the functionals in § 4 *a* (ii) vanish, respectively. For example, it follows that

$$\begin{aligned}\tilde{H}_5&[\phi, \alpha, \zeta, h, U_i] \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \tilde{\rho}_{\mathrm{f}} [-\tfrac{1}{2}(\phi_{,i} + \alpha\zeta_{,i})(\phi_{,i} + \alpha\zeta_{,i}) - \phi_{,t} - \alpha\zeta_{,t} - g x_j \delta_{3j}]\, \mathrm{d}\Omega \right. \\ &\qquad \left. + \int_{\Gamma_v} \tilde{\rho}_{\mathrm{f}} \hat{v}_\eta (\phi + \hat{\alpha}\zeta)\, \mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i]\, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i \, \mathrm{d}S \right\} \mathrm{d}t \end{aligned} \tag{4.17}$$

is obtained from functional H_6 in equation (4.11) and

$$\begin{aligned}\tilde{H}_{6p}&[p, \phi, \alpha, \zeta, h, U_i] \\ &= \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{f}}} \tilde{\rho}_{\mathrm{f}} [-\tfrac{1}{2}(\phi_{,i} + \alpha\zeta_{,i})(\phi_{,i} + \alpha\zeta_{,i}) - \phi_{,t} - \alpha\zeta_{,t} - g x_j \delta_{3j}]\, \mathrm{d}\Omega \right. \\ &\qquad \left. + \int_{\Gamma_v} \tilde{\rho}_{\mathrm{f}} \hat{v}_\eta (\phi + \hat{\alpha}\zeta)\, \mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i]\, \mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i \, \mathrm{d}S \right\} \mathrm{d}t \end{aligned} \tag{4.18}$$

from H_{7p} in equation (4.13).

The introduction of the incompressible condition excludes the equation of fluid motion, expressed in equation (3.28), from the stationary conditions of the associated functionals. This is because the velocity field v_i of an incompressible flow can be solved independently of the pressure p. The latter being determined from the dynamic equation after the evaluation of the velocity v_i through the variation of the functionals.

(b) *Fluid flow assumed irrotational*

In this situation, the fluid field velocity is given by

$$v_i = \phi_{,i}, \tag{4.19}$$

corresponding to the case $s\beta_{,i} = 0 = \alpha\zeta_{,i}$ in equation (3.21). Furthermore, the internal energy e of the fluid depends only on the fluid density ρ_{f}, whereas the enthalpy ψ depends only on pressure p. Therefore, the variational principles for an irrotational fluid flow are derived as special cases of the variational principles for the rotational examples given in § 4 *a*. These variational principles are described as follows.

(i) *Compressible fluid*

From functional H_9 it is concluded that amongst all the admissible solid displacement U_i satisfying the strain–displacement relations in equation (3.3), the velocity–displacement relations in equation (3.4), the displacement boundary condition in equation (3.8) and the time instant conditions (3.48), as well as the admissible fluid field arguments ρ_{f}, v_i and ϕ satisfying equations the first equation of (3.37) and (3.47) and the function h describing the free surface disturbance, the actual motion satisfying the governing equations in (3.1), (3.7), (3.14), (4.19), (3.29), (3.31), (3.33), (3.36), (3.39), (3.41) and (3.42) makes the 5-argument functional

$$\begin{aligned}\mathit{\Pi}_5[\rho_{\mathrm{f}}, v_i, \phi, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{f}}} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - e - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} \right) \mathrm{d}\mathit{\Omega} + \int_{\mathit{\Gamma}_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta \phi \,\mathrm{d}\mathit{\Gamma} \right\} \mathrm{d}t \\ - \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i] \,\mathrm{d}\mathit{\Omega} - \int_{S_T} \hat{T}_i U_i \,\mathrm{d}S \right\} \mathrm{d}t.\end{aligned} \tag{4.20}$$

stationary, if the constitutive relations expressed in equations (3.5), (3.6) and the second equation of (3.11) are satisfied.

Remark 4.5. If the representation expressed in equation (4.19) is considered a constraint condition on the functional $\mathit{\Pi}_5$, then the 4-argument (ρ_{f}, ϕ, h, U_i) functional

$$\begin{aligned}\mathit{\Pi}_4[\rho_{\mathrm{f}}, \phi, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{f}}} \rho_{\mathrm{f}} [-\frac{1}{2} \phi_{,i}\phi_{,i} - \phi_{,t} - e - g x_j \delta_{3j}] \,\mathrm{d}\mathit{\Omega} + \int_{\mathit{\Gamma}_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta \phi \,\mathrm{d}\mathit{\Gamma} \right\} \mathrm{d}t \\ - \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i] \,\mathrm{d}\mathit{\Omega} - \int_{S_T} \hat{T}_i U_i \,\mathrm{d}S \right\} \mathrm{d}t\end{aligned} \tag{4.21}$$

is derived. The variational constraint and stationary conditions of this functional are those of the functional $\mathit{\Pi}_5$ except that equation (4.19) is now treated as a variational constraint condition and v_i is replaced by $\phi_{,i}$ in the stationary conditions.

Remark 4.6. A repeat of the argument used to derive H_{10p} in equation (4.6) and H_{9p} in equation (4.7) allows functionals $\mathit{\Pi}_5$ and $\mathit{\Pi}_4$ to be replaced by functionals $\mathit{\Pi}_{6p}$ and $\mathit{\Pi}_{5p}$, respectively, having pressure p as an additional argument. That is,

$$\begin{aligned}&\mathit{\Pi}_{6p}[p, \rho_{\mathrm{f}}, v_i, \phi, h, U_i] \\ &\quad = \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{f}}} \rho_{\mathrm{f}} \left(\tfrac{1}{2} v_j v_j - \psi + \frac{p}{\rho_{\mathrm{f}}} - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} \right) \mathrm{d}\mathit{\Omega} + \int_{\mathit{\Gamma}_v} \hat{\rho}_{\mathrm{f}} \hat{v}_\eta \phi \,\mathrm{d}\mathit{\Gamma} \right\} \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \left\{ \int_{\mathit{\Omega}_{\mathrm{S}}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i] \,\mathrm{d}\mathit{\Omega} - \int_{S_T} \hat{T}_i U_i \,\mathrm{d}S \right\} \mathrm{d}t\end{aligned} \tag{4.22}$$

and

$$\Pi_{5p}[p,\rho_{\mathrm{f}},\phi,h,U_i] = \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\rho_{\mathrm{f}}\left[-\tfrac{1}{2}\phi_{,i}\phi_{,i}-\phi_{,t}-\psi+\frac{p}{\rho_{\mathrm{f}}}-gx_j\delta_{3j}\right]\mathrm{d}\Omega + \int_{\Gamma_v}\hat{\rho}_{\mathrm{f}}\hat{v}_\eta\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t - \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}[A(E_{ij})-B(V_i)-U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T}\hat{T}_iU_i\,\mathrm{d}S\right\}\mathrm{d}t. \tag{4.23}$$

The variational constraint conditions and the stationary conditions of these two functionals are those derived for the functionals Π_5 in (4.20) and Π_4 in (4.21), respectively.

Remark 4.7. By repeating the derivation of functional H_{14} in equation (4.8) to functional Π_{6p}, the following 10-argument (p, ρ_{f}, v_i, ϕ, h, U_i, σ_{ij}, E_{ij}, P_i, V_i) functional is obtained:

$$\begin{aligned}\Pi_{10}[p,\rho_{\mathrm{f}},v_i,\phi,h,U_i,\sigma_{ij},E_{ij},P_i,V_i] &= \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\rho_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j-\psi+\frac{p}{\rho_{\mathrm{f}}}-\ gx_j\delta_{3j}-\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)\mathrm{d}\Omega\right.\\ &\quad\left.+\int_{\Gamma_v}\hat{\rho}_{\mathrm{f}}\hat{v}_\eta\phi\,\mathrm{d}\Gamma+\int_{\Gamma_\phi}\rho_{\mathrm{f}}v_i\eta_i(\phi-\hat{\phi})\,\mathrm{d}\Gamma\right\}\mathrm{d}t\\ &\quad-\int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}[A(E_{ij})-B(V_i)-U_i\hat{F}_i+P_i(V_i-U_{i,t})\right.\\ &\quad-\sigma_{ij}(E_{ij}-\tfrac{1}{2}(U_{i,j}+U_{j,i}+U_{k,i}U_{k,j}))]\,\mathrm{d}\Omega\\ &\quad\left.-\int_{S_T}\hat{T}_iU_i\,\mathrm{d}S-\int_{S_U}\tau_{ij}\nu_j(U_i-\hat{U}_i)\,\mathrm{d}S\right\}\mathrm{d}t.\end{aligned} \tag{4.24}$$

The only variational constraint conditions on this functional are the variational conditions at times t_1 and t_2 given in equations (3.47) and (3.48). Its stationary conditions include all the governing equations of the fluid–structure dynamic interaction problem expressed in equations (3.1)–(3.8), the second equation of (3.12), (3.14), (4.19), (3.29), (3.31), (3.33), (3.36), the first equation of (3.37), (3.39), (3.41) and (3.42).

Remark 4.8. The variational constraint conditions at times t_1 and t_2 given in equations (3.47) and (3.48) can also be released (see Xing & Price 1992).

(ii) *Incompressible fluid*

The substitution of $\rho_{\mathrm{f}}=\tilde{\rho}_{\mathrm{f}}$ and $e=\psi-p/\rho_{\mathrm{f}}\equiv 0$ in § 4 *b* (i) produces the functional descriptions of an incompressible fluid. That is, functional Π_5 in equation (4.20) (or Π_{6p} in equation (4.22)), Π_4 in equation (4.21) (or Π_{5p} in equation (4.23)) and Π_{10} in equation (4.24) reduce to the respective forms

$$\tilde{\Pi}_4[v_i,\phi,h,U_i] = \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{f}}}\tilde{\rho}_{\mathrm{f}}\left(\tfrac{1}{2}v_jv_j-gx_j\delta_{3j}-\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)\mathrm{d}\Omega+\int_{\Gamma_v}\tilde{\rho}_{\mathrm{f}}\hat{v}_\eta\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t - \int_{t_1}^{t_2}\left\{\int_{\Omega_{\mathrm{S}}}[A(E_{ij})-B(V_i)-U_i\hat{F}_i]\,\mathrm{d}\Omega-\int_{S_T}\hat{T}_iU_i\,\mathrm{d}S\right\}\mathrm{d}t, \tag{4.25}$$

$$\tilde{\Pi}_3[\phi, h, U_i] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_\mathrm{f}} \tilde{\rho}_\mathrm{f}[-\tfrac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - g x_j \delta_{3j}]\,\mathrm{d}\Omega + \int_{\Gamma_v} \tilde{\rho}_\mathrm{f} \hat{v}_\eta \phi \,\mathrm{d}\Gamma \right\} \mathrm{d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_\mathrm{S}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T} \hat{T}_i U_i \,\mathrm{d}S \right\} \mathrm{d}t \tag{4.26}$$

and

$$\begin{aligned}\tilde{\Pi}_8[v_i, \phi, h, U_i, \sigma_{ij}, E_{ij}, P_i, V_i] &= \int_{t_1}^{t_2} \left\{ \int_{\Omega_\mathrm{f}} \tilde{\rho}_\mathrm{f} \left(\tfrac{1}{2} v_j v_j - g x_j \delta_{3j} - \frac{\mathrm{D}\phi}{\mathrm{D}t} \right) \mathrm{d}\Omega \right. \\ &\quad \left. + \int_{\Gamma_v} \tilde{\rho}_\mathrm{f} \hat{v}_\eta \phi \,\mathrm{d}\Gamma + \int_{\Gamma_\phi} \tilde{\rho}_\mathrm{f} v_i \eta_i (\phi - \hat{\phi})\,\mathrm{d}\Gamma \right\} \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \left\{ \int_{\Omega_\mathrm{S}} [A(E_{ij}) - B(V_i) - U_i \hat{F}_i + P_i (V_i - U_{i,t}) \right. \\ &\quad - \sigma_{ij}(E_{ij} - \tfrac{1}{2}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}))]\,\mathrm{d}\Omega \\ &\quad \left. - \int_{S_T} \hat{T}_i U_i \,\mathrm{d}S - \int_{S_U} \tau_{ij} \nu_j (U_i - \hat{U}_i)\,\mathrm{d}S \right\} \mathrm{d}t. \end{aligned} \tag{4.27}$$

(c) *Discussions*

The previously developed variational principles for nonlinear dynamical fluid–solid interaction systems are based on the concept of Hamilton's principle. They reduce to functionals describing the dynamics of either solid or fluid treated separately or fluid–structure interaction; this is now briefly discussed.

(i) *Fluid domain Ω_f and its boundary Γ_v excluded*

The functionals expressed in equations (4.1), (4.5)–(4.7), (4.10)–(4.13), (4.16)–(4.18), (4.20)–(4.23), (4.25), (4.26) reduce to the principle of potential energy in solid mechanics (see, for example, Green & Zerna 1954; Washizu 1982) and the functionals given in equations (4.14), (4.24), (4.27) reduce to the dynamical form of Hu–Washizu principle in solid mechanics (see, for example, Washizu 1982; Xing 1984).

(ii) *Solid domain Ω_s and its boundary S_T excluded*

On the assumptions of neglecting gravity potential and that all variations are taken to vanish on the boundary as adopted by Seliger & Whitham (1968), the functional in equation (4.1) reduces to

$$H_9[\rho_\mathrm{f}, v_i, \phi, s, \beta, \alpha, \zeta] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_\mathrm{f}} \rho_\mathrm{f} \left(\tfrac{1}{2} v_j v_j - e - \frac{\mathrm{D}\phi}{\mathrm{D}t} - s\frac{\mathrm{D}\beta}{\mathrm{D}t} - \alpha \frac{\mathrm{D}\zeta}{\mathrm{D}t} \right) \mathrm{d}\Omega \right\} \mathrm{d}t, \tag{4.28}$$

where the integrand is the Lagrangian density, i.e. pressure as shown by Seliger & Whitham (1968).

The associated form of the functional in equation (4.26) for a two-dimensional incompressible inviscid fluid flow assumed irrotational produces the variational principle developed by Luke (1967) accounting for variable boundaries, of which a dynamical equivalent was given by Miles (1977).

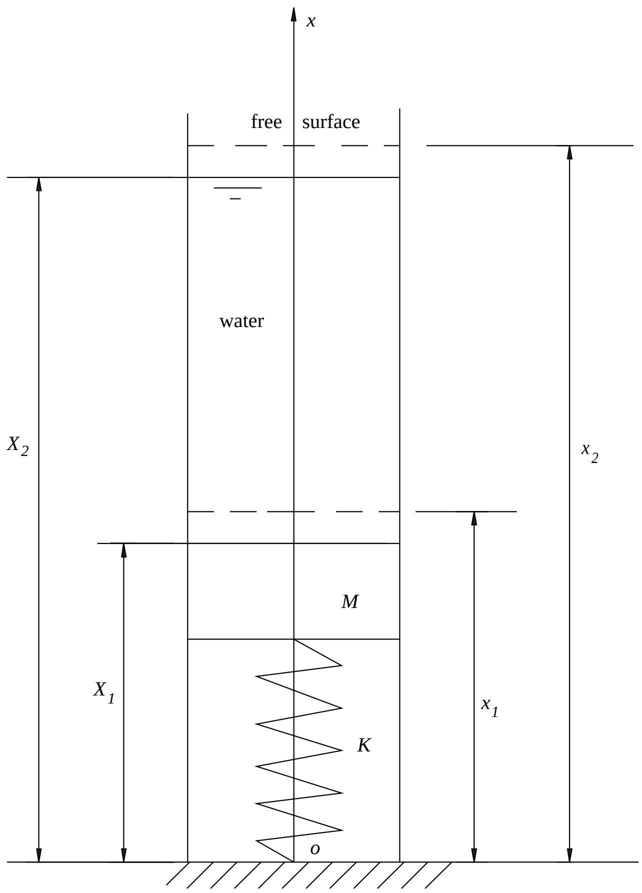

Figure 3. A one-dimensional water–mass–spring interaction system

(iii) *Special cases for fluid–solid interaction*

The functional Π_4 expressed in equation (4.21) is similar to the one presented by Kock & Olson (1991) except that the global conservation of fluid mass is included in their functional. Since this conservation relationship can be obtained through the equation of continuity in the functional, the present analysis shows that this additional requirement is a redundancy within the functional.

(iv) *Linear models*

Based on the assumption of linearity, the variational principles developed for nonlinear problems reduce to the corresponding principles for linear fluid–solid interaction problems (see, for example, Xing 1988; Morand & Ohayon 1995).

5. Examples of application

The two examples chosen are of an elementary nature but they illustrate the application of the variational principles to dynamic interaction problems without the necessity of becoming engaged in numerical analysis as would be the case if more realistic examples were examined.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(a) A one-dimensional water–mass–spring interaction problem

Figure 3 illustrates the one-dimensional fluid–structure interaction system under examination. It consists of a water column of height $L = X_2 - X_1$ with free surface, a piston of mass M and of unit sectional area and a spring of stiffness K. The water is assumed incompressible, of density $\tilde{\rho}_\mathrm{f} = 1$, and the fluid flow irrotational. In equilibrium, $x = X_1$ and $x = X_2$ denote the bottom and top positions of the water column which at any time t during motion move to positions x_1 and x_2, respectively. If X_0 represents the position of the top face of the piston when the spring is in its non-stretched state, the potential energy of the spring in its compressed state is $\frac{1}{2}K(x_1 - X_0)^2$.

For this problem, the functional $\tilde{\Pi}_3$ expressed in equation (4.26) takes the form

$$\tilde{\Pi}_{3e}[\phi, x_2, x_1] = \int_{t_1}^{t_2}\int_{x_1}^{x_2}[-\tfrac{1}{2}\phi_{,x}\phi_{,x} - \phi_{,t} - gx]\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\left[\tfrac{1}{2}K(x_1 - X_0)^2 - \tfrac{1}{2}M\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2 + Mgx_1\right]\mathrm{d}t. \tag{5.1}$$

The boundaries at x_1 and x_2 are moving points and therefore the results expressed in equation (2.32) are required when taking the variation of this functional. This gives

$$\begin{aligned}\delta^{(\mathrm{fs})}\tilde{\Pi}_{3e} &= \bar{\delta}\int_{t_1}^{t_2}\int_{x_1}^{x_2}[-\tfrac{1}{2}\phi_{,x}\phi_{,x} - \phi_{,t} - gx]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\delta\int_{t_1}^{t_2}\left[\tfrac{1}{2}K(x_1 - X_0)^2 - \tfrac{1}{2}M\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2 + Mgx_1\right]\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_{x_1}^{x_2}[-\phi_{,x}\bar{\delta}\phi_{,x} - \bar{\delta}\phi_{,t}]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\left[K(x_1 - X_0)\delta x_1 - M\frac{\mathrm{d}x_1}{\mathrm{d}t}\delta\frac{\mathrm{d}x_1}{\mathrm{d}t} + Mg\delta x_1\right]\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2}\{[\tfrac{1}{2}\phi_{,x}\phi_{,x} + \phi_{,t} + gx]|_{x=x_1}\bar{\delta}x_1 - [\tfrac{1}{2}\phi_{,x}\phi_{,x} + \phi_{,t} + gx]|_{x=x_2}\bar{\delta}x_2\}\,\mathrm{d}t.\end{aligned} \tag{5.2}$$

The application of equation (2.17), Green's theorem and the conditions $\delta x_1(t_1) = 0 = \delta x_1(t_2)$ and $\bar{\delta}\phi(t_1) = 0 = \bar{\delta}\phi(t_2)$ to this equation gives

$$\begin{aligned}\delta^{(\mathrm{fs})}\tilde{\Pi}_{3e} = \int_{t_1}^{t_2}\Bigg\{&\int_{x_1}^{x_2}\phi_{,xx}\bar{\delta}\phi\,\mathrm{d}x \\ &+ \left[\tfrac{1}{2}\phi_{,x}^2(x_1,t) + \phi_{,t}(x_1,t) + gx_1 - K(x_1 - X_0) - M\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} - Mg\right]\delta x_1 \\ &+ \left[\frac{\mathrm{d}x_2}{\mathrm{d}t} - \phi_{,x}(x_2,t)\right]\bar{\delta}\phi(x_2,t) - \left[\frac{\mathrm{d}x_1}{\mathrm{d}t} - \phi_{,x}(x_1,t)\right]\bar{\delta}\phi(x_1,t) \\ &-[\tfrac{1}{2}\phi_{,x}^2(x_2,t) + \phi_{,t}(x_2,t) + gx_2]\delta x_2\Bigg\}\,\mathrm{d}t.\end{aligned} \tag{5.3}$$

Because of the independence of the variations $\bar{\delta}\phi$ in the water domain $\Omega_\mathrm{f} = (x_1, x_2)$,

δx_1, δx_2, $\overline{\delta}\phi(x_1,t)$ and $\overline{\delta}\phi(x_2,t)$, from $\delta^{(\mathrm{fs})}\hat{\Pi}_{3e}=0$ the ordinary differential equations describing the motion of the water–mass–spring interaction system are

$$\phi_{,xx}=0, \quad (x,t)\epsilon(x_1,x_2)\times(t_1,t_2), \tag{5.4}$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}-\phi_{,x}(x_1,t)=0, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t}-\phi_{,x}(x_2,t)=0, \tag{5.5}$$

$$\tfrac{1}{2}\phi^2_{,x}(x_2,t)+\phi_{,t}(x_2,t)+gx_2=0, \tag{5.6}$$

$$M\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+K(x_1-X_0)+Mg=\tfrac{1}{2}\phi^2_{,x}(x_1,t)+\phi_{,t}(x_1,t)+gx_1. \tag{5.7}$$

Here, equation (5.4) represents Laplace's equation, the second equation of (5.5) and equation (5.6) represent the free surface boundary conditions and the first equation of (5.5) and equation (5.7) represent the interaction coupling conditions.

A solution satisfying equation (5.4) is

$$\phi=a(t)x+b(t), \quad \phi_{,x}=a(t), \quad \phi_{,t}=\frac{\mathrm{d}a}{\mathrm{d}t}x+\frac{\mathrm{d}b}{\mathrm{d}t}, \tag{5.8}$$

where $a(t)$ and $b(t)$ are two arbitrary time functions, with $a(t)$ representing a velocity. Furthermore, from equations (5.5), (5.6) and (5.7), it follows that

$$x_1=\int_{t_1}^{t}a(t)\,\mathrm{d}t+X_1, \quad x_2=\int_{t_1}^{t}a(t)\,\mathrm{d}t+X_2, \tag{5.9}$$

$$gx_2+\frac{\mathrm{d}a}{\mathrm{d}t}x_2+\frac{\mathrm{d}b}{\mathrm{d}t}+\tfrac{1}{2}a^2=0 \tag{5.10}$$

and

$$M\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+K(x_1-X_0)+Mg=gx_1+\frac{\mathrm{d}a}{\mathrm{d}t}x_1+\frac{\mathrm{d}b}{\mathrm{d}t}+\tfrac{1}{2}a^2. \tag{5.11}$$

The substitution of equation (5.10) into equation (5.11) gives

$$M\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+K(x_1-X_0)+Mg=\left(g+\frac{\mathrm{d}a}{\mathrm{d}t}\right)(x_1-x_2)=-\left(g+\frac{\mathrm{d}a}{\mathrm{d}t}\right)L, \tag{5.12}$$

and from equation (5.9) it follows that

$$M\frac{\mathrm{d}a}{\mathrm{d}t}+K\left(\int_{t_1}^{t_2}a(t)\,\mathrm{d}t+X_1-X_0\right)+(M+L)g+\frac{\mathrm{d}a}{\mathrm{d}t}L=0. \tag{5.13}$$

In the static equilibrium state,

$$K(X_1-X_0)+(M+L)g=0, \tag{5.14}$$

and equation (5.13) reduces to

$$(M+L)\frac{\mathrm{d}^2a}{\mathrm{d}t^2}+Ka=0, \tag{5.15}$$

where $a(t)$ denotes a velocity. This equation represents the dynamic equation of the system, in which the mass of the incompressible water acts as an additional mass to the piston.

(*b*) *An externally forced one-dimensional compressible gas–mass–spring dynamic interaction problem*

In this example, a one-dimensional compressible gas-structure dynamic interaction system excited by an external force is examined. As shown in figure 4, this system consists of a horizontal gas column of unit sectional area, a piston of mass M and a nonlinear spring K. The positions x_0, x_1 represent the interaction interface in static equilibrium and at instant t during motion. If X_0 denotes the position of the interaction interface relating to the unstretched spring, the elastic force of the spring is given by $K(x_1 - X_0)^2$. At initial time $t_1 = 0$, the system is in its static equilibrium position with gas pressure p_0, density ρ_0 and is excited by an external force $\hat{F}(t)$ applied to the piston.

(i) *Dynamic equations derived from functional Π_{3e}*

If the static position x_0 is chosen as a reference point of the potential energy of the spring, the potential of the spring is determined by

$$A(x_1) = \int_{x_0}^{x_1} K(x - X_0)^2 \, \mathrm{d}x. \tag{5.16}$$

In this example, the potential energy due to gravity can be neglected and the associated functional given in equation (4.21) takes the form

$$\Pi_{3e}[\rho_{\mathrm{f}}, \phi, x_1] = \int_0^{t_2} \left\{ \int_0^{x_1} \rho_{\mathrm{f}}(-\tfrac{1}{2}\phi_{,x}\phi_{,x} - \phi_{,t} - e) \, \mathrm{d}x - \left[A - \tfrac{1}{2}M \left(\frac{\mathrm{d}x_1}{\mathrm{d}t} \right)^2 - x_1 \hat{F} \right] \right\} \mathrm{d}t. \tag{5.17}$$

By taking the variation of this functional $\hat{\Pi}_{3e}$ and using the second equation of (3.11) and equation (5.16), it follows that

$$\begin{aligned} \delta^{(\mathrm{fs})} \hat{\Pi}_{3e} = \int_0^{t_2} \Bigg\{ \int_0^{x_1} & \left[\overline{\delta}\rho_{\mathrm{f}} \left(-\tfrac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - e - \frac{p}{\rho_{\mathrm{f}}} \right) - \rho_{\mathrm{f}}(\overline{\delta}\phi_{,t} + \phi_{,x}\overline{\delta}\phi_{,x}) \right] \mathrm{d}x \\ & - [\rho_{\mathrm{f}}(\tfrac{1}{2}\phi_{,i}\phi_{,i} + \phi_{,t} + e)]|_{x_1} \delta x_1 \\ & - [K(x_1 - X_0)^2 - \hat{F}]\delta x_1 + M \frac{\mathrm{d}x_1}{\mathrm{d}t} \delta \frac{\mathrm{d}x_1}{\mathrm{d}t} \Bigg\} \mathrm{d}t. \end{aligned} \tag{5.18}$$

The application of Green's theorem, equation (2.17) and conditions $\delta x_1(0) = 0 = \delta x_1(t_2)$ and $\delta\phi(0) = 0 = \delta\phi(t_2)$ gives

$$\begin{aligned} \delta^{(\mathrm{fs})} \hat{\Pi}_{3e} = \int_0^{t_2} \Bigg\{ \int_0^{x_1} & \left[\overline{\delta}\rho_{\mathrm{f}} \left(-\tfrac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - e - \frac{p}{\rho_{\mathrm{f}}} \right) + (\rho_{f,t} + (\rho_{\mathrm{f}}\phi_{,x})_{,x})\overline{\delta}\phi \right] \mathrm{d}x \\ & - \left[(\rho_{\mathrm{f}}(\tfrac{1}{2}\phi_{,i}\phi_{,i} + \phi_{,t} + e))|_{x_1} + K(x_1 - X_0)^2 - \hat{F} + M \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} \right] \delta x_1 \\ & + \left[\rho_{\mathrm{f}} \left(\frac{\mathrm{d}x_1}{\mathrm{d}t} - \phi_{,x} \right) \overline{\delta}\phi \right] \Bigg|_{x_1} + (\rho_{\mathrm{f}}\phi_{,x}\overline{\delta}\phi)|_0 \Bigg\} \mathrm{d}t. \end{aligned} \tag{5.19}$$

Because of the independence of the variations δx_1, $\overline{\delta}\rho_{\mathrm{f}}$ in the fluid domain $(0, x_1)$, $\overline{\delta}\phi$ in the fluid domain $(0, x_1)$ and at points x_1 and $x = 0$, from $\delta^{(\mathrm{fs})}\hat{\Pi}_{3e} = 0$, the following set of equations describing the dynamical problem illustrated in figure 4 are derived:

$$M \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + K(x_1 - X_0)^2 = \hat{F} - [\rho_{\mathrm{f}}(\tfrac{1}{2}\phi_{,i}\phi_{,i} + \phi_{,t} + e)]|_{x_1}, \quad (x, t)\epsilon(0, x_1) \times (0, t_2), \tag{5.20}$$

$$\tfrac{1}{2}\phi_{,i}\phi_{,i} + \phi_{,t} + e + \frac{p}{\rho_{\mathrm{f}}} = 0, \quad (x, t)\epsilon(0, x_1) \times (0, t_2), \tag{5.21}$$

$$\rho_{f,t} + (\rho_{\mathrm{f}}\phi_{,x})_{,x} = 0, \quad (x, t)\epsilon(0, x_1) \times (0, t_2), \tag{5.22}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

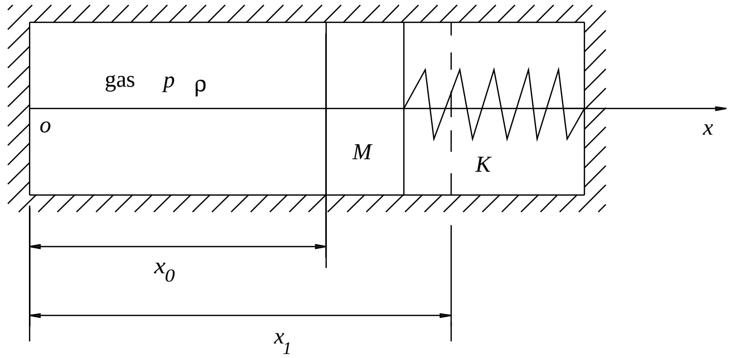

Figure 4. A one-dimensional compressible gas–mass–spring interaction system

$$\rho_{\mathrm{f}}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}-\phi_{,x}\right)=0, \quad x=x_1, \tag{5.23}$$

$$\rho_{\mathrm{f}}\phi_{,x}=0, \quad x=0. \tag{5.24}$$

Here, equation (5.20) represents the dynamic interaction mechanism within this forced system, equation (5.21) Bernoulli's equation, equation (5.22) the compressibility of the gas and equations (5.23) and (5.24) boundary conditions.

(ii) *Approximate solution derived by functional* Π_{3e}

Here, an approximation method is described to solve the posed problem. Let us, for simplification purposes, treat the density of the gas as a function of time t only. This implies the gas remains homogeneous throughout the motion. The internal energy e is determined from equation (3.13) and the velocity potential takes the following approximate form:

$$\phi(x,t)=\tfrac{1}{2}q(t)x^2, \quad \phi_{,t}=\tfrac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}t}x^2, \quad \phi_{,x}=qx, \tag{5.25}$$

where $q(t)$ is an unknown function of time to be determined. It is easy to check that the selected solution $\phi(x,t)$ in equation (5.25) satisfies the boundary condition (5.24). However, this is not a requirement of the constraints for functional Π_{3e}. The substitution of this solution into functional (5.17) and its integration with respect to x gives

$$\Pi_{3e}[\rho_{\mathrm{f}},q,x_1]=\int_0^{t_2}\left\{\rho_{\mathrm{f}}\left(-\tfrac{1}{6}q^2x_1^3-\tfrac{1}{6}\frac{\mathrm{d}q}{\mathrm{d}t}x_1^3-ex_1\right)-\left[A-\tfrac{1}{2}M\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2-x_1\hat{F}\right]\right\}\mathrm{d}t. \tag{5.26}$$

Through an integration by parts and the time instant conditions $\delta q(0)=0=\delta q(t_2)$ and $\delta x_1(0)=0=\delta x_1(t_2)$, the variation of this functional takes the form

$$\begin{aligned}\delta^{(\mathrm{fs})}\Pi_{3e}[\rho_{\mathrm{f}},q,x_1]=\int_0^{t_2}&\left\{\bar{\delta}\rho_{\mathrm{f}}\left(-\tfrac{1}{6}q^2x_1^3-\tfrac{1}{6}\frac{\mathrm{d}q}{\mathrm{d}t}x_1^3-ex_1-\frac{p}{\rho_{\mathrm{f}}}x_1\right)\right.\\&-\left(\tfrac{1}{3}qx_1^3-\tfrac{1}{2}x_1^2\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)\delta q-\left[\rho_{\mathrm{f}}\left(\tfrac{1}{2}q^2x_1^2+\tfrac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}t}x_1^2+e\right)\right.\\&\left.\left.+K(x_1-X_0)^2-\hat{F}+M\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}\right]\delta x_1\right\}\mathrm{d}t.\end{aligned} \tag{5.27}$$

Therefore, from $\delta^{(\mathrm{fs})}\Pi_{3e} = 0$ and the independence of the variations $\overline{\delta}\rho_{\mathrm{f}}$, δq and δx_1, the set of equations obtained are

$$M\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + K(x_1 - X_0)^2 = \hat{F} - \rho_{\mathrm{f}} e - \tfrac{1}{2}\rho_{\mathrm{f}}\left(q^2 + \frac{\mathrm{d}q}{\mathrm{d}t}\right)x_1^2, \tag{5.28}$$

$$\tfrac{1}{6}\rho_{\mathrm{f}}\left(q^2 + \frac{\mathrm{d}q}{\mathrm{d}t}\right)x_1^2 + \rho_{\mathrm{f}} e + p = 0, \tag{5.29}$$

$$\tfrac{1}{3}q x_1 - \tfrac{1}{2}\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0. \tag{5.30}$$

These equations form a system of ordinary differential equation with arguments $x_1(t)$, $q(t)$ and $\rho_{\mathrm{f}}(t)$ supplanting the system of partial differential dynamic equations given in equations (5.20)–(5.24). Through equation (3.13) and the imposition of realistic initial conditions, a solution can be derived subject to the imposed limitation of the model.

6. Conclusion

A family of variational principles describing the dynamical behaviour of nonlinear fluid–solid interaction systems is developed and each member discussed. The mathematical model is based on the concept of Hamilton's principle and the fundamental equations of continuum mechanics. The variational principles admit variable boundaries and their forms reflect the assumptions inherent to the model (i.e. compressible fluid, flow rotational, etc.). Variational principles obtained previously to determine the dynamic characteristics of an elastic body in solid mechanics, fluid flows in fluid dynamics and fluid–solid interaction are derived from the presented variational principles by the introduction of the appropriate assumptions and/or boundaries.

The development of the variational principles creates a rigorous theoretical foundation to build computational models such as finite element methods and numerical schemes of study (see, for example, Zienkiewicz & Taylor 1989, 1991; Xing *et al.* 1996) to solve complex nonlinear fluid–structure dynamic interaction problems in engineering.

J.T.X. expresses his deep appreciation to NSFC for supporting the related research in China.

Appendix A. Nomenclature

A	function of strain energy per unit volume of solid
B	function of kinetic energy per unit volume of solid
e	internal energy per unit mass of fluid
e_{ijk}	permutation tensor
E_{ij}	Green's strain tensor
$\hat{f}_i$	vector of body force per unit mass of fluid
$\hat{F}_i$	vector of body force per unit volume of solid
g	acceleration due to gravity
h	unknown function of (x_1, x_2, x_3, t) describing motion on the free surface Γ_{f}
J	Jacobian of a transformation
N	translation velocity of a curved surface in space

p	pressure field of fluid
P_i, $\boldsymbol{P}$	momentum vector of solid, $\boldsymbol{P} = (P_1, P_2, P_3)$
s	entropy per unit mass of fluid
S	surface of solid domain Ω_{S} $(= S_U \cup S_T \cup \Sigma)$
S_U	part of S with prescribed displacement $\hat{U}_i$
S_T	part of S with prescribed traction $\hat{T}_i$
t	time variable
t_1	initial time of motion
t_2	final time of motion
T	temperature
$\hat{T}_i$	traction vector prescribed on surface S_T of solid
u_i, $\boldsymbol{u}$	displacement vector of continuum, $\boldsymbol{u} = (u_1, u_2, u_3)$
δu_η	normal component of δx_i on free surface Γ_{f} $(= \delta x_i \eta_i)$
U_i, $\boldsymbol{U}$	displacement vector of solid, $\boldsymbol{U} = (U_1, U_2, U_3)$
$\hat{U}_i$	displacement vector prescribed on surface S_U of solid
δU_ν	normal component of δU_i on interaction boundary Σ $(= \delta U_i \nu_i)$
δU_ξ	tangent component of δU_i on interaction boundary Σ $(= \delta U_i \xi_i)$
v_i	velocity field of fluid
V_i, $\boldsymbol{V}$	velocity vector of solid, $\boldsymbol{V} = (V_1, V_2, V_3)$
W_i, $\boldsymbol{W}$	acceleration vector of solid, $\boldsymbol{W} = (W_1, W_2, W_3)$
x_i, $\boldsymbol{x}$	spatial coordinates, $\boldsymbol{x} = (x_1, x_2, x_3)$
X_i, $\boldsymbol{X}$	material coordinates, $\boldsymbol{X} = (X_1, X_2, X_3)$
$\alpha(\boldsymbol{x}, t)$	a field function chosen as a material coordinate of the continuum
β	temperature displacement
γ	ratio of the constant specific heats of an ideal gas
Γ	surface of fluid domain Ω_{f} $(= \Gamma_{\mathrm{f}} \cup \Gamma_v \cup \Gamma_\phi \cup \Sigma)$
Γ_{f}	free surface of fluid
Γ_v	part of Γ with prescribed normal velocity of fluid $\hat{v}_\eta$
Γ_ϕ	part of Γ with prescribed velocity potential $\hat{\phi}$ and mass density $\hat{\rho}_{\mathrm{f}}$
Γ_{F}	surface of fluid domain Ω_{F}
Γ_{M}	surface of fluid domain Ω_{M}
δ_{ij}	Kronecker delta tensor
ε	parameter of variation $(-1 < \varepsilon < 1)$
ϵ	symbol to denote the meaning 'belonging to'
ζ	Clebsch potential corresponding to α
η_i	unit vector along outer normal of Γ
θ	transmission velocity of a curved surface in space
ν_i	unit vector along outer normal of S
ξ_i	unit vector along tangent direction of Σ
ρ	mass density of continuum
ρ_{f}	mass density of fluid
$\tilde{\rho}_{\mathrm{f}}$	prescribed constant mass density of incompressible fluid
ρ_{S}	mass density of solid
σ_{ij}	second Kirchhoff stress tensor
Σ	fluid–solid interaction interface between Ω_{f} and Ω_{S}
τ_{ij}	Piola stress tensor
υ	specific volume of fluid $(= 1/\rho_{\mathrm{f}})$
ϕ	velocity potential of fluid

ψ	enthalpy per unit mass of fluid
ω_i	vorticity of velocity field
$\Omega_{\rm f}$	fluid domain
$\hat{\Omega}_{\rm f}$	closed fluid domain ($=\Omega_{\rm f}\cup\Gamma$)
$\Omega_{\rm F}$	fixed domain in space
$\Omega_{\rm M}$	material domain in continuum
$\Omega_{\rm S}$	solid domain
$\hat{\Omega}_{\rm S}$	closed solid domain ($=\Omega_{\rm S}\cup S$)
i,j,k	indices ($=1,2,3$) of a tensor, obeying the summation convention
$\mathrm{d}()/\mathrm{d}t$	time derivative of () ($=\dot{()}$)
$\mathrm{D}()/\mathrm{D}t$	material derivative of ()
grad()	gradient of ()
$()_{,t}$	$=\partial()/\partial t$
$()_{,i}$	$=\partial()/\partial x_i$ or $=\partial()/\partial X_i$
$\bar{\delta}()$	local variation of ()
$\delta()$	material variation of ()
$\delta^{(\mathrm{fs})}()$	variation of () ($=\bar{\delta}()$ for fluid but $\delta()$ for solid)
$\sim$	denotes equality for terms of order 1 relative to ε

References

Bathe, K. J., Nitikitpaiboon, C. & Wang, X. 1995 A mixed displacement-based finite element formulation for acoustic fluid–structure interaction. *Comput. Struct.* **56**, 225–237.

Courant, R. & Hilbert, D. 1962 *Methods of mathematical physics*, vol. 2. New York: Interscience.

Ecer, A. & Akay H. U. 1983 Applications of variational principles in computing rotational flows. In *Recent advances in numerical methods in fluids: advances in computational transonics* (ed. W. G. Habashi). Swansea: Pineridge.

Ecer, A., Rout, R. K. & Ward, P. 1983 Investigation of solution of Navier–Stokes equations using a variational formulation. *Int. J. Numer. Methods Fluids* **3**, 23–31.

Forsyth, A. R. 1890 *Theory of differential equations*, vol. 1. Cambridge University Press.

Fung, Y. C. 1977 *A first course in continuum mechanics*. Englewood Cliffs, NJ: Prentice-Hall.

Galper, A. & Miloh, T. 1995 Dynamic equations of motion for a rigid or deformable body in an arbitrary non-uniform potential flow field. *J. Fluid Mech.* **295**, 91–120.

Gelfand, I. M. & Fomin, S. V. 1963 *Calculus of variations*. Englewood Cliffs, NJ: Prentice-Hall.

Green, A. E. & Adkins, J. E. 1960 *Large elastic deformations (and nonlinear continuum mechanics)*. Oxford University Press.

Green, A. E. & Zerna, W. 1954 *Theoretical elasticity*. Oxford University Press.

Hunter, S. C. 1976 *Mechanics of continuous media*. New York: Wiley.

Ikegawa, M. & Washizu, K. 1973 Finite element method applied to analysis of flow over a spillway crest. *Int. J. Numer. Methods Engng* **6**, 179–189.

Kock, E. & Olson, L. 1991 Fluid–solid interaction analysis by the finite element method—a variational approach. *Int. J. Numer. Methods Engng* **31**, 463–491.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Lin, C. C. 1963 *Liquid helium*. Proc. Int. School of Physics, Course XXI. New York: Academic.

Liu, W. K. & Uras, R. A. 1988 Variational approach to fluid–structure interaction with sloshing. *Nucl. Engng Des.* **106**, 69–85.

Luke, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

Malvern, L. E. 1970 *An introduction to the mechanics of a continuum media*. Englewood Cliffs, NJ: Prentice-Hall.

McIver, D. B. 1973 Back to virtual power. In *Variational methods in engineering* (Department of Civil Engineering, University of Southampton), vol. 1, pp. 57–68. Southampton University Press.

Miles, J. L. 1977 Hamilton's principles for surface waves. *J. Fluid Mech.* **83**, 153–158.

Miloh, T. 1977 Hamilton's principles, Lagrange's method, and ship motion theory. *J. Ship Research* **28**, 229–237.

Morand, H. J.-P. & Ohayon, R. 1995 *Fluid structure interaction*. New York: Wiley.

Oden, J. T. 1972 *Finite elements of nonlinear continua*. New York: McGraw-Hill.

Oden, J. T. & Reddy, J. N. 1976 *Variational methods in theoretical mechanics*. Berlin: Springer.

Rainey, R. C. T. 1989 A new equation for calculating wave loads on offshore structures. *J. Fluid Mech.* **204**, 295–324.

Seliger, R. L. & Whitham, G. B. 1968 Variational principles in continuum mechanics. *Proc. R. Soc. Lond.* A **305**, 1–25.

Serrin, J. 1959 Mathematical principles of classical fluid mechanics. In *Encyclopedia of physics* (ed. S. Flugge), vol. VIII/1, pp. 125–263. Berlin: Springer.

Taub, A. H. 1949 On Hamilton's principle for perfect compressible fluids. *Proc. Symp. Appl. Math., N.Y.* **1**, 148–157.

Truesdell, C., 1966 *The elements of continuum mechanics*. Berlin: Springer.

van Daalen, E. F. G., Broeze, J. & van Groesen, E. 1992 Variational principles and conservation laws in the derivation of radiation boundary conditions for wave equations. *Math. Comp.* **58**, 55–71.

Ward, P., Desai, R., Kebede, W. & Ecer, A. 1988 A variational finite-element formulation for 3-dimensional incompressible flows. In *Numerical methods for fluid dynamics III* (ed. K. W. Morton & M. T. Baines), ch. 46, pp. 403–409. Oxford University Press.

Washizu, K. 1982 *Variational methods in elasticity and plasticity*. Sydney: Pergamon.

Woods, L. C. 1975 *The thermodynamics of fluid systems*. Oxford University Press.

Xing, J. T. 1984 Some theoretical and computational aspects of finite element method and substructure-subdomain technique for dynamic analysis of the coupled fluid–solid interaction problems—variational principles for elastodynamics and linear theory of micropolar elasticity with their applications to dynamic analysis. Ph.D. dissertation, Department of Engineering Mechanics, Qinghua University, Beijing, China. (In Chinese.)

Xing, J. T. 1986 A study on finite element method and substructure–subdomain technique for dynamic analysis of coupled fluid–solid interaction problems. *Acta Mech. Solida Sinica* **4**, 329–337. (In Chinese.)

Xing, J. T. 1988 Two variational formulations for dynamic analysis of coupled fluid–solid interaction problems with linearized free surface wave considered. *Acta Aero. Astro. Sinica* **9**, 568–571. (In Chinese.)

Xing, J. T. & Price, W. G. 1991 A mixed finite element method for the dynamic analysis of coupled fluid–solid interaction problems. *Proc. R. Soc. Lond.* A **433**, 235–255.

Xing, J. T. & Price, W. G. 1992 Some generalized variational principles for conservative holonomic dynamical systems. *Proc. R. Soc. Lond.* A **436**, 331–344.

Xing, J. T. & Price, W. G. 1996 General theorems and generalized variational principles for nonlinear elastodynamics. Report no. 99, Department of Ship Science, University of Southampton.

Xing, J. T., Price, W. G. & Du, Q. H. 1996 Mixed finite element substructure–subdomain methods for the dynamic analysis of coupled fluid–solid interaction problems. *Phil. Trans. R. Soc. Lond.* A **354**, 259–295.

Zienkiewicz, O. C. & Taylor, R. L. 1989 *The finite element method*, 4th edn, vol. 1. New York: McGraw-Hill.

Zienkiewicz, O. C. & Taylor, R. L. 1991 *The finite element method*, 4th edn, vol. 2. New York: McGraw-Hill.

Received 30 April 1996; accepted 17 September 1996